

FINITENESS OF A_n -EQUIVALENCE TYPES OF GAUGE GROUPS

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ABSTRACT. Let B be a finite CW complex and G a compact connected Lie group. We show that the number of gauge groups of principal G -bundles over B is finite up to A_n -equivalence for $n < \infty$. As an example, we give a lower bound of the number of A_n -equivalence types of gauge groups of principal $SU(2)$ -bundles over S^4 .

1. INTRODUCTION

Let G be a topological group and P be a principal G -bundle over a space B . A G -equivariant map $P \rightarrow P$ covering the identity map on B is called an *automorphism* of P . The *gauge group* $\mathcal{G}(P)$ of P is the topological group consisting of all automorphisms of P .

Let us consider the following problem: how many homotopy types of $\mathcal{G}(P)$ do exist for fixed B and G ? For $B = S^4$ and $G = SU(2)$, Kono [11] has shown that there exist only six homotopy types of $\mathcal{G}(P)$. More generally, for any finite CW complex B and any compact connected Lie group G , Crabb and Sutherland show that there exist only finitely many homotopy types of $\mathcal{G}(P)$ in [4]. Moreover, they show that the number of homotopy types of $\mathcal{G}(P)$ as H -spaces, namely H -types of $\mathcal{G}(P)$ is also finite.

Stasheff considered the concept of an A_n -map [18] between topological monoids. An H -map between topological monoids is exactly an A_2 -map between them. Here, we can consider the more general problem: how many A_n -equivalence types, which we also refer as A_n -types, of gauge groups do exist for fixed B and G ? Especially, let us consider this problem for any finite complex B and any compact connected Lie group G in case when n is finite. For $n = 1$ or 2 , as stated above, Crabb and Sutherland show that it is finite. We will give a more general result for $n \geq 3$.

Theorem 1.1. *Let B be a finite complex and G be a compact connected Lie group. As P ranges over all principal G -bundles with base B , the number of A_n -equivalence types of $\mathcal{G}(P)$ is finite in case when n is finite.*

In general, the number of A_∞ -types of $\mathcal{G}(P)$ is not finite (§9).

If we regard G as a left G -space by the adjoint action $(g, x) \mapsto gxg^{-1}$, the bundle $\text{aut } P = P \times_G G$ with fibre G associated to P is a fibrewise topological group, namely, a group object in the comma category $\mathbf{Top} \downarrow B$ and the space $\Gamma(\text{aut } P)$ of all sections of $\text{aut } P$ and $\mathcal{G}(P)$ are isomorphic as topological groups. So to prove Theorem 1.1, it is sufficient to show that the number of *fibrewise A_n -equivalence types* of $\text{aut } P$ is finite.

In §2, we review the terminology of fibrewise homotopy theory. In §3, we review associahedra and multiplihedra. In §4, we review the definitions of *fibrewise A_n -spaces* and *fibrewise A_n -maps* and see some fundamental properties. These notions are the fibrewise versions of A_n -spaces [17] and A_n -maps [7] respectively. In §5, we show the classification theorem for fibrewise A_n -spaces with fibre A_n -equivalent to some fixed A_n -space. In §6 and 7, we treat the fibrewise localization of fibrewise A_n -spaces. We see that the fibrewise rationalizations of automorphism bundles are trivial. In §8, we complete the proof of Theorem 1.1. In §9, we show the existence of a counterexample to Theorem 1.1, if we disregard the condition “ n is finite” in Theorem 1.1. In §10, we make a new observation of the gauge groups of principal $SU(2)$ -bundles over S^4 . Especially, we give a lower bound of the number of the A_n -types of such gauge groups.

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2. FIBREWISE SPACES AND FIBREWISE POINTED SPACES

We follow the terminology introduced in [3] used in describing the fibrewise homotopy theory.

Let E and B be spaces. We say E is a *fibrewise space* over B if a map $\pi : E \rightarrow B$, called the *projection* of E , is given. For each $b \in B$, we denote $E_b := \pi^{-1}(b)$ and call E_b the *fibre* over b . The space B itself is regarded as

a fibrewise space over B with projection given by the identity map. Every space can be seen as a fibrewise space over a point with the unique projection.

From now on, we always assume that all fibrewise spaces are Hurewicz fibrations.

Let $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B$ be fibrewise spaces. If a map $f : E \rightarrow E'$ satisfies $\pi' f = \pi$, then f is called a *fibrewise map* over B . We denote I as the unit interval $[0, 1]$. Regard $I \times E$ as a fibrewise space over B with projection given by composing π with the second projection $I \times E \rightarrow E$. Let $f, g : E \rightarrow E'$ be fibrewise maps. A fibrewise map $h : I \times E \rightarrow E'$ is called a *fibrewise homotopy* between f and g if $h|_{0 \times E} = f$ and $h|_{1 \times E} = g$. If there exists such h , then f and g are said to be *fibrewise homotopic*.

For fibrewise spaces $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B$, the fibre product $E \times_B E'$ of E and E' is defined by

$$E \times_B E' = \{ (e, e') \in E \times E' \mid \pi(e) = \pi'(e') \}.$$

We denote the i -fold fibre product of E by $E^{\times_B i}$.

Fibrewise mapping space $\text{map}_B(E, E')$ between fibrewise spaces E and E' over the same base B is the following set with appropriate topology:

$$\text{map}_B(E, E') = \coprod_{b \in B} \text{Map}(E_b, E'_b),$$

where $\text{Map}(E_b, E'_b)$ is the set of all continuous maps $E_b \rightarrow E'_b$. This space is naturally a fibrewise space over B .

In the remainder of this paper, we work in the category of fibrewise compactly-generated spaces, which is introduced in [8]. For a fibrewise space E over B , if E and B are compactly-generated, then E is fibrewise compactly-generated. Let E, E' and E'' be fibrewise spaces over B and $f : E \times_B E' \rightarrow E''$ be a function with $\pi_{E''} f = \pi_{E \times_B E'}$, where $\pi_{E''} : E'' \rightarrow B$ and $\pi_{E \times_B E'} : E \times_B E' \rightarrow B$ are the projections. Then we define the function $f' : E \rightarrow \text{map}_B(E', E'')$ by $f'(x)(y) = f(x, y)$.

Proposition 2.1 ((Proposition(5.6) in [8])). *Assume E and E' are locally sliceable [3]. Then f is continuous if and only if f' is continuous.*

If $E \rightarrow B$ is a Hurewicz fibration and B is a CW complex, then E is locally sliceable. This criterion is sufficient for our later use.

A *fibrewise pointed space* E over B is a fibrewise space $E \xrightarrow{\pi} B$ with a section σ of π . Here, each fibre E_b is regarded as a pointed space with basepoint $\sigma(b)$. The fibrewise space B over B is regarded as a fibrewise pointed space with the section given by the identity map. Every pointed space is a fibrewise pointed space over a point.

Let $B \xrightarrow{\sigma} E \xrightarrow{\pi} B$ and $B \xrightarrow{\sigma'} E' \xrightarrow{\pi'} B$ be fibrewise pointed spaces. A fibrewise map $f : E \rightarrow E'$ is a *fibrewise pointed map* if $f\sigma = \sigma'$. Moreover, if f is a homeomorphism, we say f is a *fibrewise pointed topological equivalence*.

3. REVIEW OF ASSOCIAHEDRA AND MULTIPLIHEDRA

We will review and construct associahedra and multiplihedra using the W -construction in [2] since we will use the result of Boardman and Vogt in [2]. These constructions are equivalent to the original construction in [17] respectively in [7]. This fact is also stated in [5]. Here we remark that we do not consider the degeneracy maps of associahedra and multiplihedra. Because we do not need to assume that an A_n -form has a strict unit. Details of this will be explained in the next section.

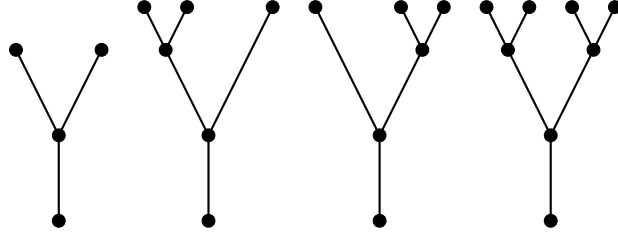
First, we recall basic definitions about trees.

Definition 3.1. *A planted plane tree (see §2 of [10]) τ is said to be an unpainted tree if each vertex in τ is not connected to exactly two edges. For a planted plane tree τ , let $V(\tau)$ be the set of all vertices in τ and define the subset*

$$L(\tau) = \{ v \in V(\tau) \mid v \text{ is not the root and is connected to only one edge.} \}.$$

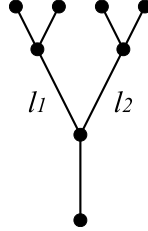
A vertex contained in $L(\tau)$ is called a leaf and a vertex which is not the root or a leaf is called an internal vertex. Similarly, the edge whose boundary contains the root is called the root edge, an edge whose boundary contains a leaf is called a leaf edge and an edge which is not the root edge or a leaf edge is called an internal edge. Moreover, if each internal vertex in an unpainted tree τ is connected to just three edges, τ is said to be binary.

For a planted plane tree τ , we give each edge the direction toward the root. Then, each internal vertex in τ has some incoming edges and the unique outgoing edge.

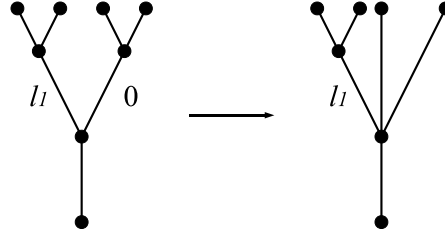


Remark 1. For an integer $n \geq 2$, a binary unpainted tree with n leaves has just $n - 2$ internal edges.

Definition 3.2. Let τ be a planted plane tree. If each internal edge of τ is labeled by an element of $I = [0, 1]$, τ is said to be a metric tree.

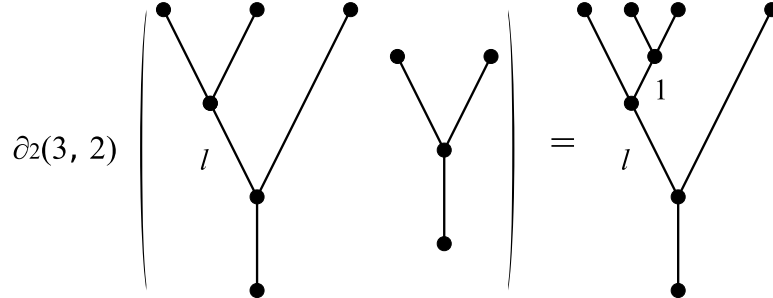


Denote the set of all binary unpainted trees with n leaves by T_n . The space $T\mathcal{A}_n$ consisting of all binary unpainted metric trees with n leaves is topologized by $T\mathcal{A}_n = T_n \times I^{n-2}$. Now we define an equivalence relation in $T\mathcal{A}_n$. For $\rho, \tau \in T\mathcal{A}_n$, remove all internal edges with length 0 and unite the end vertices of each removed edges.



Then we have the new (in general, not binary) unpainted metric trees $R(\rho), R(\tau)$. We say ρ and τ are equivalent if $R(\rho) = R(\tau)$. This relation defines the quotient space \mathcal{K}_n of $T\mathcal{A}_n$. For example, \mathcal{K}_2 is a point, \mathcal{K}_3 is a line segment and \mathcal{K}_4 is a pentagon.

Let us define the grafting map. For $\rho \in T\mathcal{A}_r, \tau \in T\mathcal{A}_t$ and an integer $1 \leq k \leq r$, we can make the new binary unpainted metric tree $\partial_k(r, t)(\rho, \tau)$ by identifying the root edge of τ and the k -th leaf edge of ρ , where this identification is compatible with the direction of edges and the length of the new internal edge is $1 \in I$.



This defines a continuous map $\partial_k(r, s) : \mathcal{K}_r \times \mathcal{K}_t \rightarrow \mathcal{K}_{r+t-1}$. These grafting maps satisfy the following conditions:

$$\partial_j(p, r+t-1)(1 \times \partial_k(r, t)) = \partial_{j+k-1}(p+r-1, t)(\partial_j(p, r) \times 1),$$

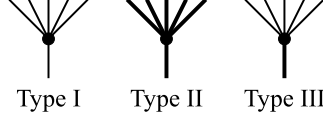
$$\text{for } k < j, \quad \partial_{j+r-1}(p+r-1, t)(\partial_k(p, r) \times 1) = \partial_k(p+t-1, r)(\partial_j(q, t) \times 1)(1 \times T)$$

where T transposes the factors. Let \mathcal{L}_n be the union of images of these grafting maps in \mathcal{K}_n . Then \mathcal{K}_n is homeomorphic to the cone $C\mathcal{L}_n$ of \mathcal{L}_n . Therefore, these spaces are Stasheff's associahedra in [17]. This implies that there is a homeomorphism $(\mathcal{K}_n, \mathcal{L}_n) \simeq (D^{n-2}, S^{n-3})$ for $n \geq 3$.

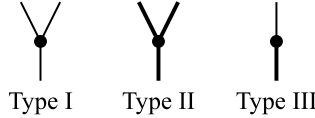
In the above construction, if we replace unpainted metric trees by *painted metric trees*, then we can construct multiplihedra. We consider two types of edges of painted trees, say *unpainted edges* and *painted edges*.

Definition 3.3. A painted tree is a planted plane tree with edges labeled by the set { painted, unpainted } satisfying the following conditions:

- (i) each of its internal vertices is one of the following types:
 Type I: all incoming edges and the outgoing edge are unpainted,
 Type II: all incoming edges and the outgoing edge are painted,
 Type III: all incoming edges are unpainted and the outgoing edge is painted,
 where the number of incoming edges of a vertex of type I or II is greater than 1,
 (ii) all leaf edges are unpainted while the root edge is painted.



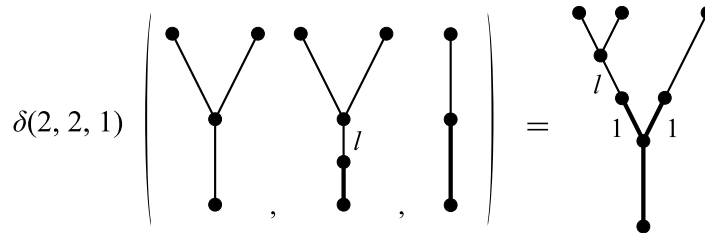
A painted tree is called binary if the number of incoming edges of each internal vertex of type I or II is 2 and the number of incoming edges of each internal vertex of type III is 1.



For painted metric trees, we consider the following two types of grafting maps.

Like unpainted metric trees, we can graft an unpainted metric tree onto a painted metric tree. For a painted metric tree ρ with r leaves, an unpainted metric tree τ with t leaves and an integer $1 \leq k \leq r$, we can make the new painted metric tree $\delta_k(r, t)(\rho, \tau)$ by identifying the root edge of τ and the k -th leaf edge of ρ , where this identification is compatible with the direction of edges, the length of the new internal edge is $1 \in I$ and each edge in $\delta_k(r, t)(\rho, \tau)$ which comes from τ is unpainted.

Conversely, we can graft painted metric trees onto an unpainted metric tree. For an unpainted metric tree τ with t leaves and painted metric trees ρ_1, \dots, ρ_t such that each ρ_i has r_i leaves, $\delta(t, r_1, \dots, r_t)(\tau, \rho_1, \dots, \rho_t)$ is the painted metric tree constructed by identifying the root edge of ρ_i and the i -th leaf edge of τ for each i , where this identification is compatible with the direction of edges, the lengths of the new internal edges are $1 \in I$ and each edge which comes from τ is painted.



It is not quite appropriate for our purpose to consider all painted metric trees, so we consider a new class of painted metric trees. For an painted metric tree τ , let $M(\tau)$ be the length of the longest internal edge in τ . If τ has no internal vertices, then define $M(\tau) = 0$.

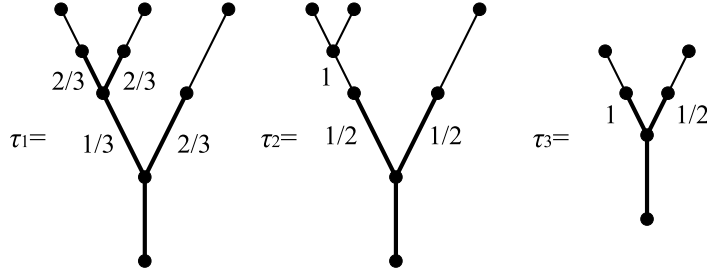
Definition 3.4. Level-trees are painted metric trees inductively defined as follows:

- (I) If a painted metric tree τ satisfies the condition $M(\tau) = 0$, then τ is a level-tree.
 (II) Let τ be a painted metric tree with n internal edges such that $M(\tau) > 0$. Define a new painted metric tree $\tilde{\tau}$ as follows:

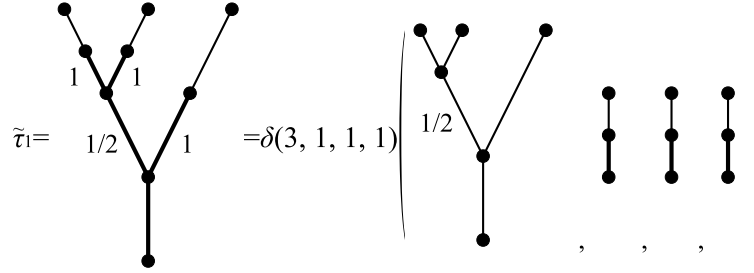
- (i) $\tilde{\tau} = \tau$ as painted trees,
 (ii) for each internal edge $e \in \tilde{\tau}$, if the length of e in τ is $\ell \in I$, then the length of e in $\tilde{\tau}$ is $\ell/M(\tau) \in I$.

If $\tilde{\tau} = \delta_k(r, s)(\rho, \sigma)$ for some level-tree ρ and unpainted metric tree σ or $\tilde{\tau} = \delta(r, s_1, \dots, s_r)(\rho', \sigma'_1, \dots, \sigma'_r)$ for some unpainted metric tree ρ and level-trees $\sigma'_1, \dots, \sigma'_r$, then τ is a level-tree. Here we remark that the number of internal edges in each of such level-trees $\rho, \sigma'_1, \dots, \sigma'_r$ is less than n and can define level-trees inductively.

Example 1. Consider painted metric trees τ_1, τ_2, τ_3 as the following figure.

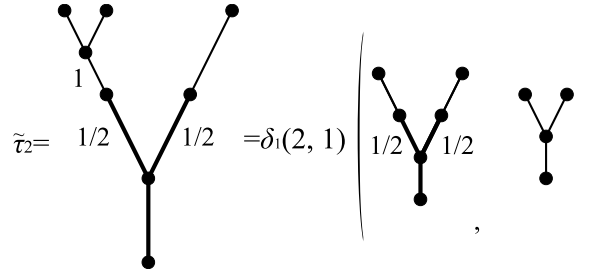


(i) Since $M(\tau_1) = 2/3$, $\tilde{\tau}_1$ is described as the following figure.



Hence τ_1 is a level-tree.

(ii) Since $M(\tau_2) = 1$, $\tilde{\tau}_2 = \tau_2$ and $\tilde{\tau}_2$ is decomposed as follows.



As easily checked, the painted metric tree in the right hand side is a level-tree. Then τ_2 is also a level-tree.

(iii) Since $M(\tau_3) = 1$, $\tilde{\tau}_3 = \tau_3$. The painted metric tree $\tilde{\tau}_3$, however, can not be described as any grafting of some painted metric trees and unpainted metric trees. Thus τ_3 is not a level-tree.

Let TM_n be the set consisting of all binary level-trees with n -leaves. We topologize TM_n by the embedding

$$TM_n \subset \bigcup_{\substack{\tau: \text{ a binary painted tree} \\ \text{ with } n \text{ leaves}}} \{\tau\} \times I^{\times N(\tau)},$$

where $N(\tau)$ denotes the number of internal edges in τ . Just like unpainted metric trees, we consider reduction of edges with length 0. For $\tau \in TM_n$, remove all internal edges with length 0 and unite the end vertices of each removed edges. We denote this new level-tree by $R(\tau)$. We say ρ and $\tau \in TM_n$ are equivalent if $R(\rho) = R(\tau)$ as painted metric trees. Let \mathcal{J}_n be the quotient space of TM_n by this relation. For example, \mathcal{J}_1 is a point, \mathcal{J}_2 is a line segment and \mathcal{J}_3 is a hexagon.

The grafting constructions define the continuous maps $\delta_k(r, t) : \mathcal{J}_r \times \mathcal{K}_t \rightarrow \mathcal{J}_{r+t-1}$ and $\delta(t, r_1, \dots, r_t) : \mathcal{K}_t \times \mathcal{J}_{r_1} \times \dots \times \mathcal{J}_{r_t} \rightarrow \mathcal{J}_{r_1+\dots+r_t}$. These maps satisfy the following conditions:

$$\begin{aligned} & \delta_j(p, r+t-1)(1 \times \partial_k(r, t)) = \delta_{j+k-1}(p+r-1, t)(\delta_j(p, r) \times 1), \\ \text{for } k < j, & \delta_{j+r-1}(p+r-1, t)(\delta_k(p, r) \times 1) = \delta_k(p+t-1, r)(\delta_j(p, t) \times 1)(1 \times T) \\ & \delta_{p_1+\dots+p_{j-1}+k}(p, r)(\delta(t, p_1, \dots, p_t) \times 1) \\ \text{for } p_1 + \dots + p_t = p \text{ and } 1 \leq k \leq r_j, & \\ & = \delta(t, p_1, \dots, p_{j-1}, p_j+r-1, p_{j+1}, \dots, p_t)(1^{\times j} \times \delta_k(p_j, r) \times 1^{\times q-j})T_j \\ & \delta(r+t-1, p_1, \dots, p_{r+t-1})(\partial_k(r, t) \times 1^{\times r+t-1}) \\ & = \delta(r, p_1, \dots, p_{k-1}, p_k + \dots + p_{k+t-1}, p_{k+t}, \dots, p_{r+t-1})(1^{\times k} \times \delta(t, p_k, \dots, p_{k+t-1}) \times 1^{\times r-k})T'_k, \end{aligned}$$

where the maps T_j and T'_k are defined by $T_j(\tau, \pi_1, \dots, \pi_t, \rho) = (\tau, \pi_1, \dots, \pi_j, \rho, \pi_{j+1}, \dots, \pi_t)$ and $T'_k(\rho, \tau, \pi_1, \dots, \pi_{r+t-1}) = (\rho, \pi_1, \dots, \pi_{k-1}, \tau, \pi_k, \dots, \pi_{r+t-1})$. Let \mathcal{H}_n be the union of images of these grafting maps in \mathcal{J}_n . Then \mathcal{J}_n is homeomorphic to the cone $C\mathcal{H}_n$ of \mathcal{H}_n . Therefore, \mathcal{J}_n is the n -th multiplihedron constructed by Iwase and Mimura in [7]. This implies that there is a homeomorphism $(\mathcal{J}_n, \mathcal{H}_n) \simeq (D^{n-1}, S^{n-2})$ for $n \geq 2$.

We compare the above constructions of associahedra and multiplihedra with the W -construction in [2]. Let \mathcal{A} be the PRO of semigroups and \mathcal{L}_1 be the linear category $0 \rightarrow 1$. In other words, \mathcal{A} is the PRO such that $\mathcal{A}(0, 1)$ is empty and $\mathcal{A}(n, 1)$ is a point for $n \geq 1$. From the above construction, $W\mathcal{A}(1, 1)$ is a point, $W\mathcal{A}(n, 1) = \mathcal{K}_n$ for $n \geq 2$, $LW(\mathcal{A} \otimes \mathcal{L}_1)(n^0, 1^1) = \mathcal{J}_n$ for $n \geq 1$ and the compositions in these PROs are compatible with our grafting maps.

4. FIBREWISE A_n -SPACES AND FIBREWISE A_n -MAPS

We shall review fibrewise A_n -spaces and fibrewise A_n -maps. For any fibrewise space E over B , we will consider $\mathcal{K}_i \times E$ and $\mathcal{J}_i \times E$ as a fibrewise spaces over B .

Definition 4.1. A fibrewise space E over B is called a fibrewise A_n -space (without unit) if a family of fibrewise maps $\{m_i : \mathcal{K}_i \times E^{\times_{B^i}} \rightarrow E\}_{i=2}^n$, called a fibrewise A_n -form of E , is given and satisfies the following condition:

$$\text{for } x_l \in E_b, \rho \in \mathcal{K}_r, \tau \in \mathcal{K}_t, \quad m_i(\partial_k(r, t)(\rho, \tau); x_1, \dots, x_i) = m_r(\rho; x_1, \dots, x_{k-1}, m_t(\tau; x_k, \dots, x_{k+t-1}), x_{k+t}, \dots, x_i)$$

Let $(E, \{m_i\})$ and $(E', \{m'_i\})$ be fibrewise A_n -spaces over B . A fibrewise map $f : E \rightarrow E'$ is called a fibrewise A_n -map if there exists a family of fibrewise maps $\{f_i : \mathcal{J}_i \times E^{\times_{B^i}} \rightarrow E'\}_{i=1}^n$, called a fibrewise A_n -form of f , such that

$$f_1 = f$$

for $x_l \in E_b, \rho \in \mathcal{J}_r, \tau \in \mathcal{K}_t$,

$$f_i(\delta_k(r, t)(\rho, \tau); x_1, \dots, x_i) = f_r(\rho; x_1, \dots, x_{k-1}, m_t(\tau; x_k, \dots, x_{k+t-1}), x_{k+t}, \dots, x_i)$$

for $x_l \in E_b, \rho_s \in \mathcal{J}_{r_s}, \tau \in \mathcal{K}_t$,

$$f_i(\delta(t, r_1, \dots, r_t)(\tau, \rho_1, \dots, \rho_t); x_1, \dots, x_i) = m'_i(\tau; f_{r_1}(\rho; x_1, \dots, x_{r_1}), \dots, f_{r_t}(\rho_t; x_{r_1+\dots+r_{t-1}+1}, \dots, x_i))$$

A fibrewise A_n -equivalence is a fibrewise A_n -map which is also a fibrewise homotopy equivalence. If there exists a fibrewise A_n -equivalence between two fibrewise A_n -spaces, then they are said to be fibrewise A_n -equivalent.

In particular, a fibrewise A_n -space over a point is called an A_n -space. The terms such as A_n -map etc. are similarly defined.

Remark 2. (i) We do not suppose the existence of strict or homotopy units of fibrewise A_n -spaces. Because the author does not know whether the universal fibrewise A_n -space $E_n(G)$ constructed in §5 has a fibrewise homotopy unit or not. Every fibrewise pointed space is a fibrewise A_∞ -space by a trivial fibrewise A_∞ -form in the above sense. But we do not have any difficulties in showing Theorem 1.1 with our definition of fibrewise A_n -forms. Because our definition of A_n -maps between topological monoids agrees with the definition in [18].

(ii) We do not require that fibrewise A_n -spaces are fibrewise pointed and fibrewise A_n -forms preserve the basepoint in each fibre. But even if we do, the following arguments are verified analogously. In that situation, some spaces are replaced by “based” ones. For example, the universal fibration is replaced by the fibrewise pointed universal fibration and the mapping space $\text{Map}(G^n, G)$ in §5 is replaced by $\text{Map}^*(G^n, G)$, which is the subspace of $\text{Map}(G^n, G)$ consisting of maps $G^n \rightarrow G$ preserving the basepoint.

(iii) Fibrewise A_n -spaces are exactly multiplicative functors from $Q^n W\mathcal{A}$ (see Remark 3.19 in [2]) to the category of fibrewise spaces in the sense of [2]. Similarly, fibrewise A_n -maps are fibrewise level-tree maps between fibrewise $Q^n W\mathcal{A}$ -spaces. Here, we define $Q^n LW(\mathcal{A} \otimes \mathcal{L}_1)$ as the PRO-subcategory of $LW(\mathcal{A} \otimes \mathcal{L}_1)$ generated by the morphisms of $LW(\mathcal{A} \otimes \mathcal{L}_1)(\underline{i}, k)$, $\underline{i} : [r] \rightarrow \text{ob } \mathcal{L}_1$ with $r \leq n$ and a fibrewise level-tree maps between fibrewise $Q^n W\mathcal{A}$ -spaces E and E' means a multiplicative functor from $Q^n LW(\mathcal{A} \otimes \mathcal{L}_1)$ to the category of fibrewise spaces of which the restrictions to $d^1 Q^n W\mathcal{A}$ and $d^0 Q^n W\mathcal{A}$ give fibrewise $Q^n W\mathcal{A}$ -spaces E and E' respectively.

Example 2. (i) Every topological monoid is an A_∞ -space. More generally, every fibrewise topological monoid is a fibrewise A_∞ -space. Here, a fibrewise pointed space $B \xrightarrow{\sigma} E \xrightarrow{\pi} B$ (σ is a section of π) with a fibrewise map $m : E \times_B E \rightarrow E$ is a fibrewise topological monoid if $m(1 \times_B m) = m(m \times_B 1)$ and $m(\sigma(\pi(x)), x) = m(x, \sigma(\pi(x))) = x$ for each $x \in E$. If (E, m) is a fibrewise topological monoid, then the family of maps $\{m_i : \mathcal{K}_i \times E^{\times_{B^i}} \rightarrow E\}$ defined by $m_i(\rho; x_1, \dots, x_i) = m(x_1, \dots, m(x_{i-1}, x_i) \dots)$ is a fibrewise A_∞ -form of E .

(ii) An H -space is an A_2 -space. A homotopy associative H -space (G, m) together with its associating homotopy $m(1 \times m) \simeq m(m \times 1)$ is an A_3 -space.

(iii) For a pointed space X , the based loop space ΩX of X is naturally an A_∞ -space. In fact, an A_∞ -form of ΩX is constructed as follows. Let

$$P_i = \{ (t_0, t_1, \dots, t_i) \in I^{i+1} \mid 0 = t_0 < t_1 < \dots < t_i = 1 \}$$

be the space of partitions of the unit interval $I = [0, 1]$. Since P_i is contractible, one can construct an A_∞ -form $\{m_i : \mathcal{K}_i \times (\Omega X)^{\times i} \rightarrow \Omega X\}$ of ΩX such that

$$m_i(\rho; \ell_1, \dots, \ell_i)(t) = \ell_k \left(\frac{t - \omega_{k-1}^i(\rho)}{\omega_k^i(\rho) - \omega_{k-1}^i(\rho)} \right)$$

for $\rho \in \mathcal{K}_i$, $\ell_1, \dots, \ell_i \in \Omega X$, $\omega_{k-1}^i(\rho) \leq t \leq \omega_k^i$, where $\omega^i = (\omega_0^i, \omega_1^i, \dots, \omega_i^i) : \mathcal{K}_i \rightarrow P_i$. One can take this family $\{\omega^i\}$ as independent of X . Since P_i is contractible, such A_∞ -form of ΩX is unique up to homotopy of A_∞ -forms.

(iv) Every homomorphism between topological monoids is an A_∞ -map. We can consider a more general situation. Let $(E, \{m_i\})$ and $(E', \{m'_i\})$ be fibrewise A_n -spaces. A fibrewise map $f : E \rightarrow E'$ is called a fibrewise A_n -homomorphism if $m'_i(1 \times f^{\times bi}) = f m_i$ for each i , where a fibrewise A_n -form of f is constructed by forgetting the painting of trees in \mathcal{J}_r . For example, if X and Y are pointed spaces and $f : X \rightarrow Y$ is a pointed map, then the map $\Omega f : \Omega X \rightarrow \Omega Y$ is an A_∞ -homomorphism.

(v) An H -map between H -spaces is an A_2 -map.

The results by Boardman and Vogt in Chapter IV, §2 and §3 of [2] remain true even if $W\mathcal{B}$ -spaces are replaced by “fibrewise $W\mathcal{B}$ -spaces”. Hence, we have the following propositions.

Proposition 4.2 ((Corollary 4.15 in [2])). *Let E , E' and E'' be fibrewise A_n -spaces over B and $f : E \rightarrow E'$ and $g : E' \rightarrow E''$ be fibrewise A_n -maps. Then $gf : E \rightarrow E''$ is also a fibrewise A_n -map.*

Proposition 4.3 ((Corollary 4.20 in [2])). *Let E be a fibrewise space over B and E' be a fibrewise A_n -space over B . If $f : E \rightarrow E'$ is a fibrewise homotopy equivalence, then there exists a fibrewise A_n -form of E such that f is a fibrewise A_n -equivalence.*

Proposition 4.4 ((Corollary 4.21 in [2])). *Let E and E' be fibrewise A_n -spaces over B and $f : E \rightarrow E'$ be a fibrewise A_n -equivalence. Then the fibrewise homotopy inverse $g : E' \rightarrow E$ of f is also a fibrewise A_n -equivalence.*

Remark 3. (i) From Propositions 4.2 and 4.4, the fibrewise A_n -equivalence is an equivalence relation.

(ii) Boardman and Vogt have not explicitly shown the level-tree map version. However, Proposition 4.6 in [2] guarantees us the above propositions. The $Q^n W\mathcal{B}$ -space version can also be verified.

(iii) For the fibrewise pointed version, we need to assume the fibrewise homotopy extension property of the sections of fibrewise pointed spaces. This property corresponds to well-pointedness of pointed spaces. For the pointed version, see Chapter 5, §5 in [2].

5. THE CLASSIFICATION THEOREM FOR FIBREWISE A_n -SPACES

Let B' be a space, $f : B' \rightarrow B$ be a map and E be a fibrewise A_n -space over B . The pull-back f^*E of E by f is naturally a fibrewise A_n -space.

Let E and E' be fibrewise A_n -spaces over B and B' respectively. We say a fibre map f covering $\bar{f} : B \rightarrow B'$ is a fibrewise A_n -map over \bar{f} if the induced map $E \rightarrow \bar{f}^*E'$ by f is a fibrewise A_n -map. In particular, $\bar{f}^*E' \rightarrow E'$ is a fibrewise A_n -map over \bar{f} .

Proposition 5.1. *Let E and E' be fibrewise A_n -spaces over B and B' respectively. If a fibrewise A_n -map $f : E \rightarrow E'$ over \bar{f} is homotopic to a fibre map $g : E \rightarrow E'$ covering $\bar{g} : B \rightarrow B'$ with a homotopy from f to g covering a homotopy from \bar{f} to \bar{g} , then g is a fibrewise A_n -map over \bar{g} .*

Proof. Take a homotopy $h : I \times E \rightarrow E'$ between f and g covering a homotopy $\bar{h} : I \times B \rightarrow B'$ between \bar{f} and \bar{g} . We construct a fibrewise A_n -form $\{h'_i : \mathcal{J}_i \times I \times E^{\times bi} \rightarrow \bar{h}^*E'\}$ of the fibrewise map $h' : I \times E \rightarrow \bar{h}^*E'$ induced by h , where $\{h'_i|_{0 \times B}\}$ is equal to the fibrewise A_n -form $\{f'_i\}$ of the fibrewise map $E \rightarrow \bar{f}^*E'$ induced by f .

Assume we have constructed h'_1, \dots, h'_{i-1} . Since E' has the homotopy lifting property and $(\mathcal{J}_i, \mathcal{H}_i) \simeq (D^{i-1}, S^{i-2})$, we can extend the fibrewise map $h'_i : ((\mathcal{J}_i \times 0) \cup (\mathcal{H}_i \times I)) \times E^{\times bi} \rightarrow \bar{h}^*E'$ defined by

$$h'_i(\rho, s; x_1, \dots, x_i) = \begin{cases} f'_i(\rho; x_1, \dots, x_i) & (s = 0) \\ h'_i(\rho'; s; x_1, \dots, x_{k-1}, m_i(\tau'; x_k, \dots, x_{k+t-1}), x_{k+t}, \dots, x_i) & (\rho = \delta_k(r, t)(\rho', \tau')) \\ m'_i(\tau'; h'_{r_1}(\rho'_1, s; x_1, \dots, x_{r_1}), \dots, h'_{r_t}(\rho'_t, s; x_{r_1+\dots+r_{t-1}+1}, \dots, x_i)) & (\rho = \delta(t, r_1, \dots, r_t)(\tau', \rho'_1, \dots, \rho'_t)) \end{cases}$$

to $\mathcal{J}_i \times I \times E^{\times_{B^i}}$, where $\{m_i\}$ and $\{m'_i\}$ are fibrewise A_n -forms of E and E' , respectively. We have constructed a desired map h_i . Therefore g is a fibrewise A_n -map over \bar{g} . \square

Remark 4. In the pointed case, since we have used the homotopy lifting property, we have to assume that E' is an ex-fibration (see [3]).

Proposition 5.2. Let E be a fibrewise A_n -space over B and B' be a space. If maps $f, g : B' \rightarrow B$ are homotopic, then f^*E and g^*E are fibrewise A_n -equivalent.

Proof. It is sufficient to prove that for a fibrewise A_n -space E over $I \times B$, $E|_{0 \times B}$ and $E|_{1 \times B}$ are fibrewise A_n -equivalent. From the homotopy lifting property of E , there exists a fibrewise map $h : I \times (E|_{0 \times B}) \rightarrow E$ over $I \times B$. From Proposition 5.1, $h|_{1 \times E} : E|_{0 \times B} \rightarrow E|_{1 \times B}$ is a fibrewise A_n -map. Since $h|_{1 \times E}$ is a fibrewise homotopy equivalence, $E|_{0 \times B}$ and $E|_{1 \times B}$ are fibrewise A_n -equivalent. \square

In the rest of this section, we will construct the classifying space for fibrewise A_n -spaces with fibres A_n -equivalent to a fixed A_n -space by the same method as in [4]. Let $(G, \{\mu_i\})$ be an A_n -space and B be a space, where both G and B has homotopy types of CW complexes. Assume every fibre of a fibrewise A_n -space is A_n -equivalent to G in the rest of this section.

Let E be a fibrewise space such that fibres of E are homotopy equivalent to G . Define

$$M_n[E] = \coprod_{b \in B} \left\{ \{m_i\} \in \prod_{i=2}^n \text{Map}(\mathcal{K}_i \times E_b^{\times i}, E_b) \mid \begin{array}{l} \{m_i\} : \text{an } A_n\text{-form of } E_b \text{ such that} \\ (E_b, \{m_i\}) \text{ and } (G, \{\mu_i\}) \text{ are } A_n\text{-equivalent} \end{array} \right\}.$$

Here, we have the projection $M_n[E] \rightarrow M_{n-1}[E]$ defined by forgetting m_n . Since the inclusion $\mathcal{L}_n \rightarrow \mathcal{K}_n$ has the homotopy extension property, this projection is a Hurewicz fibration, of which the fibres are homotopy equivalent to the component $\mathcal{Q}_0^{n-2} \text{Map}(G^{\times n}, G)$ containing the constant map $S^{n-2} \rightarrow \text{Map}(G^{\times n}, G)$ of $\mathcal{Q}^{n-2} \text{Map}(G^{\times n}, G)$ with basepoint of $\text{Map}(G^{\times n}, G)$ given by the map $(x_1, \dots, x_n) \mapsto \mu_2(x_1, \dots, \mu_2(x_{n-1}, x_n), \dots)$. Then a fibrewise space $E_n[E]$ is defined as the pull-back of E by the projection $M_n[E] \rightarrow B$. Define the fibrewise map $\tilde{m}_i : K_i \times E_n[E]^{\times_{M_n[E]} i} \rightarrow E_n[E]$ by $\tilde{m}_i(\rho; \{m_i\}, x_1, \dots, x_i) = m_i(\rho; x_1, \dots, x_i)$, then $\{\tilde{m}_i\}_{i=2}^n$ is a fibrewise A_n -form of $E_n[E]$. Using the space $M_n[E]$, we can state the Lemma 5.7 in [2] in the following form.

Lemma 5.3 ((Lemma 5.7 in [2])). Let E be a fibrewise A_n -space and $\{m_i\}, \{m'_i\} : B \rightarrow M_n[E]$ be fibrewise A_n -forms of E . Then the identity map of E is a fibrewise A_n -equivalence between $\{m_i\}$ and $\{m'_i\}$ if and only if they are homotopic as sections of $M_n[E] \rightarrow B$.

Recall that there exists the universal fibration (not *principal*) $E_1 \rightarrow M_1 = BFG$ for Hurewicz fibrations with fibres homotopy equivalent to G , where FG is the space of all self homotopy equivalences of G . For any space B with homotopy type of a CW complex, there is the bijection between the free homotopy set $[B; M_1]$ and the set of all fibrewise homotopy equivalence classes of Hurewicz fibrations with fibres homotopy equivalent to G given by pull-back of E_1 . Refer [14] for details.

We denote $M_n = M_n(G) = M_n[E_1]$ and $E_n = E_n(G) = E_n[E_1]$.

Proposition 5.4. Let E be a fibrewise A_n -space over B . Then there exists a fibrewise A_n -map $f : E \rightarrow E_n$ over $\bar{f} : B \rightarrow M_n$ such that the induced map $E \rightarrow \bar{f}^*E_n$ is a fibrewise A_n -equivalence.

Proof. Let $f_1 : E \rightarrow E_1$ be a fibre map covering $\bar{f}_1 : B \rightarrow M_1$ which induces a fibrewise homotopy equivalence $E \rightarrow \bar{f}_1^*E_1$. From Propositions 4.3 and 4.4, there exists a fibrewise A_n -form of $\bar{f}_1^*E_1$ such that $E \rightarrow \bar{f}_1^*E_1$ is a fibrewise A_n -equivalence. This fibrewise A_n -form gives a section of $\bar{f}_1^*M_n = M_n[\bar{f}_1^*E_1] \rightarrow B$. This section and f_1 gives a fibrewise A_n -map $f : E \rightarrow E_n$ over $\bar{f} : B \rightarrow M_n$ which induces fibrewise A_n -equivalence $E \rightarrow \bar{f}^*E_n$. \square

Lemma 5.5. Let E be a fibrewise A_n -space over B and $f, g : B \rightarrow M_n$ be maps such that f^*E_n and g^*E_n are fibrewise A_n -equivalent to E . Then f and g are homotopic.

Proof. Let $p : M_n \rightarrow M_1$ be the projection. Then pf and pg are homotopic. Let us denote this homotopy $h' : I \times B \rightarrow M_0$. Since p is a Hurewicz fibration, there exists a homotopy $h : I \times B \rightarrow M_n$ such that $ph = h'$ and $h_0 = f$. Take a fibrewise A_n -map $\hat{f} : E \rightarrow E_n$ over f which induces a fibrewise A_n -equivalence $E \rightarrow f^*E_n$. From the homotopy lifting property of E_n , there exists a homotopy $\hat{h} : I \times E \rightarrow E_n$ such that \hat{h} covers h and $\hat{h}_0 = \hat{f}$. Here E is fibrewise A_n -equivalent to both g^*E_n and $h_1^*E_n$. Since $ph_1 = g$, they are the same fibrewise spaces. From Propositions 4.4, 4.2 and 5.1, the identity map $h_1^*E_n \rightarrow g^*E_n$ is a fibrewise A_n -equivalence and Lemma 5.3 says that h_1 is homotopic to g . Therefore, f and g are homotopic. \square

The next proposition follows from this lemma immediately.

Proposition 5.6. *Let E and E' be fibrewise A_n -equivalent fibrewise A_n -spaces over B and $f, g : B \rightarrow M_n$ be maps. If E and E' are fibrewise A_n -equivalent to f^*E_n and g^*E_n respectively, then f and g are homotopic.*

From Proposition 5.2 and Proposition 5.6, we conclude the classification theorem.

Theorem 5.7 ((the classification theorem for fibrewise A_n -spaces)). *Let G be an A_n -space and B be a space, both of which have homotopy types of CW complexes. Fix a finite positive integer n . Let us denote $A_n(G; B)$ the set of all fibrewise A_n -equivalent classes of fibrewise A_n -spaces over B with fibres A_n -equivalent to G . Then the map $[B; M_n(G)] \rightarrow A_n(G; B)$ defined by pull-back of E_n is well-defined and bijective.*

We will construct the associated principal fibration of this universal fibration for our later calculation. Let C_n be the fibrewise space over M_n defined as the space consisting of all A_n -equivalences with its A_n -form from G to some fibre of E_n :

$$C_n = \coprod_{b \in M_n} \left\{ \{f_i\} \in \prod_{i=1}^n \text{Map}(\mathcal{J}_i \times G^{\times i}, (E_n)_b) \mid \begin{array}{l} \{f_i\} : \text{an } A_n\text{-form of an } A_n\text{-equivalence} \\ \text{from } G \text{ to } (E_n)_b \end{array} \right\}.$$

As easily checked, the projection $C_n \rightarrow M_n$ is a Hurewicz fibration. The fibres of $C_n \rightarrow M_n$ are homotopy equivalent to the space $F_{A_n}G$ consisting of all A_n -forms of self A_n -equivalences of G . It follows from the following proposition that C_n is ∞ -connected.

Proposition 5.8. *Let $(G, \{\mu_i\}_{i=2}^n)$ be an A_n -space, $(F, \{\mu_i\}_{i=2}^{n-1})$ an A_{n-1} -space and $(f, \{f_i\}_{i=1}^{n-1}) : G \rightarrow F$ an A_{n-1} -equivalence. Then the space*

$$\begin{aligned} \Phi = \{ (\mu_n, f_n) \mid (f, \{f_i\}_{i=1}^n) : (G, \{\mu_i\}_{i=2}^n) \rightarrow (F, \{\mu_i\}_{i=2}^n) \text{ is an } A_n\text{-equivalence.} \} \\ \subset \text{Map}(\mathcal{K}_n \times F^{\times n}, F) \times \text{Map}(\mathcal{J}_n \times G^{\times n}, F) \end{aligned}$$

is contractible.

Proof. For a fibrewise $Q^n W\mathcal{B}$ -space $X, Y : Q^n W\mathcal{B} \rightarrow \mathbf{FWCG}$, we say $g : Q^n W(\mathcal{B} \otimes \mathcal{L}_1) \rightarrow \mathbf{FWCG}$ is a fibrewise $Q^n \mathcal{B}$ -map from X to Y if $X = d^1 f$ and $Y = d^0 f$, where \mathbf{FWCG} denotes the category of fibrewise compactly generated spaces. As remarked in §4, similar results in Chapter IV of [2] hold for fibrewise $Q^n \mathcal{B}$ -maps. Let $\Psi = \Phi \times F$ be a fibrewise A_n -space over Φ such that the A_n -form of the fibre $\Psi_{(\mu_n, f_n)} = \{(\mu_n, f_n)\} \times F$ is $\{\mu_i\}_{i=2}^n$. Then $1 \times f : \Phi \times (G, \{\mu_i\}_{i=2}^n) \rightarrow \Psi$ is a fibrewise A_n -equivalence such that the A_n -form of $f_{(\mu_n, f_n)} : (G, \{\mu_i\}_{i=2}^n) \rightarrow \Psi_{(\mu_n, f_n)}$ is $\{f_i\}_{i=1}^n$. This fibrewise A_n -map $1 \times f : \Phi \times (G, \{\mu_i\}_{i=2}^n) \rightarrow \Psi$ extends to a fibrewise $Q^n \mathcal{A}$ -map $\hat{f} : Q^n W(\mathcal{A} \otimes \mathcal{L}_1) \rightarrow \mathbf{FWCG}$. If we fix a point $(\mu_n^0, f_n^0) \in \Phi$, the fibrewise map $1 \times (f, \{f_1, \dots, f_{n-1}, f_n^0\}) : \Phi \times (G, \{\mu_i\}_{i=2}^n) \rightarrow \Phi \times (F, \{\mu_2, \dots, \mu_{n-1}, \mu_n^0\})$ is a fibrewise A_n -equivalence and extends to a fibrewise $Q^n \mathcal{A}$ -map $\hat{f}^0 : Q^n W(\mathcal{A} \otimes \mathcal{L}_1) \rightarrow \mathbf{FWCG}$. Then the identity map $\Psi \rightarrow \Phi \times F$ gives a fibrewise $Q^n \mathcal{A}$ -map $Q^n W(\mathcal{A} \otimes \mathcal{L}_1) \rightarrow \mathbf{FWCG}$.

$$\begin{array}{ccc} \Phi \times (G, \{\mu_i\}_{i=2}^n) & \xrightarrow{\hat{f}} & \Psi \\ \downarrow \text{id} & & \downarrow \\ \Phi \times (G, \{\mu_i\}_{i=2}^n) & \xrightarrow{\hat{f}^0} & \Phi \times (F, \{\mu_2, \dots, \mu_{n-1}, \mu_n^0\}) \end{array}$$

The above diagram commutes in the sense of the section 2 in Chapter IV of [2]. Then we can construct a fibrewise $Q^n(\mathcal{A} \otimes \mathcal{L}_1)$ -map $h : Q^n W(\mathcal{A} \otimes \mathcal{L}_1 \otimes \mathcal{L}_1) \rightarrow \mathbf{FWCG}$ from \hat{f} to \hat{f}^0 whose underlying map is the identity map. Thus, from Lemma 5.7 in [2], $\hat{f}, \hat{f}^0 : Q^n W(\mathcal{A} \otimes \mathcal{L}_1) \rightarrow \mathbf{FWCG}$ are homotopic. This implies that the space Φ is contractible to a point (μ_n^0, f_n^0) . \square

6. FIBREWISE LOCALIZATION OF FIBREWISE A_n -SPACES

We introduce the fibrewise \mathcal{P} -localization of fibrewise A_n -spaces. In the following, notice our fibrewise spaces are Hurewicz fibrations.

Definition 6.1. *Let E and \bar{E} be fibrewise spaces over B and \mathcal{P} be a family of prime numbers. A fibrewise map $\ell : E \rightarrow \bar{E}$ is a fibrewise \mathcal{P} -localization if the restriction $\ell_b : E_b \rightarrow \bar{E}_b$ of ℓ to each fibre E_b is a \mathcal{P} -localization. We say a fibrewise space is fibrewise \mathcal{P} -local if each of its fibres is \mathcal{P} -local.*

Remark 5. *Fibrewise localizations have the following universal property: if E' is a fibrewise \mathcal{P} -local space and $f : E \rightarrow E'$ is a fibrewise map, then there exists a fibrewise map $\bar{f} : \bar{E} \rightarrow E'$, unique up to fibrewise homotopy, such that $\bar{f}\ell$ is fibrewise homotopic to f , where the base space of E, \bar{E} and E' has homotopy type of a CW complex.*

We quote Theorem 4.1 of [15] in the next form.

Theorem 6.2. *Let G be a nilpotent finite CW complex and $\ell : G \rightarrow G_{\mathcal{P}}$ be the \mathcal{P} -localization of G . Then, ℓ induces a map $\ell_{\#} : BFG \rightarrow BFG_{\mathcal{P}}$ which corresponds to the map $FG \rightarrow FG_{\mathcal{P}}$ induced by ℓ , which \mathcal{P} -localizes the identity component.*

Let G be a nilpotent finite CW complex and E_1 be a universal fibration of G . Then, we can construct the fibrewise \mathcal{P} -localization $E_1 \rightarrow \bar{E}_1$ of E_1 , where the classifying map of \bar{E}_1 is $\ell_{\#}$. Hence if E is a fibrewise space with fibres homotopy equivalent to G and with base B homotopy equivalent to a CW complex, then we can take the fibrewise \mathcal{P} -localization $\ell : E \rightarrow \bar{E}$ of E .

Proposition 6.3. *Let E be a fibrewise A_n -space. Then the fibrewise \mathcal{P} -localization $\ell : E \rightarrow \bar{E}$ has the fibrewise A_n -forms of \bar{E} and ℓ , both of which are unique up to homotopy through fibrewise A_n -forms. For an A_n -space G with homotopy type of a nilpotent finite complex, this fibrewise localization induces the map $M_n(G) \rightarrow M_n(G_{\mathcal{P}})$ and the following diagrams commute up to homotopy:*

$$\begin{array}{ccccccc} M_n(G) & \longrightarrow & \cdots & \longrightarrow & M_2(G) & \longrightarrow & BFG \\ \downarrow & & & & \downarrow & & \downarrow \ell_{\#} \\ M_n(G_{\mathcal{P}}) & \longrightarrow & \cdots & \longrightarrow & M_2(G_{\mathcal{P}}) & \longrightarrow & BFG_{\mathcal{P}} \\ \\ \Omega_0^{n-2} \text{Map}(G^{\times n}, G) & \longrightarrow & & \longrightarrow & M_n(G) & \longrightarrow & M_{n-1}(G) \\ \downarrow & & & & \downarrow & & \downarrow \\ \Omega_0^{n-2} \text{Map}(G_{\mathcal{P}}^{\times n}, G_{\mathcal{P}}) & \longrightarrow & & \longrightarrow & M_n(G_{\mathcal{P}}) & \longrightarrow & M_{n-1}(G_{\mathcal{P}}) \end{array}$$

where the map $\Omega_0^{n-2} \text{Map}(G^{\times n}, G) \rightarrow \Omega_0^{n-2} \text{Map}(G_{\mathcal{P}}^{\times n}, G_{\mathcal{P}})$ is the \mathcal{P} -localization.

Proof. Let $\{m_i\}$ be the fibrewise A_n -form of E . We construct fibrewise A_n -forms $\{\bar{m}_i\}$ and $\{\ell_i\}$ of \bar{E} and ℓ respectively. Let $\ell_1 = \ell$. Assume we have constructed fibrewise A_{j-1} -forms $\{\bar{m}_i\}_{i=2}^{j-1}$ and $\{\ell_i\}_{i=1}^{j-1}$. The fibrewise map $\ell'_j : (\mathcal{H}_j - \text{Int} \delta(j, 1, \dots, 1)) \times E^{\times_{Bj}} \rightarrow \bar{E}$ can be defined by using $\{m_i\}$, $\{\bar{m}_i\}$ and $\{\ell_i\}$ in a natural manner and be extended on $\mathcal{J}_j \times E^{\times_{Bj}}$. We can also define $\bar{m}_j : \mathcal{L}_j \times \bar{E}^{\times_{Bj}} \rightarrow \bar{E}$ by using $\{\bar{m}_i\}$ and $\ell'_j(\delta(j, 1, \dots, 1) \times 1)|_{\mathcal{L}_j \times E^{\times_{Bj}}} = \bar{m}_j(1 \times \ell^{\times_{Bj}})$. There exists a fibrewise map $\bar{m}'_j : \mathcal{K}_j \times \bar{E}^{\times_{Bj}} \rightarrow \bar{E}$ such that $\bar{m}'_j(1 \times \ell^{\times_{Bj}})$ is fibrewise homotopic to $\ell'_j(\delta(j, 1, \dots, 1) \times 1)$. Then \bar{m}_j is fibrewise homotopic to $\bar{m}'_j|_{\mathcal{L}_j \times \bar{E}^{\times_{Bj}}}$. Hence \bar{m}_j can be extended on $\mathcal{K}_j \times \bar{E}^{\times_{Bj}}$ and there exists $\ell_j : \mathcal{J}_j \times E^{\times_{Bj}} \rightarrow \bar{E}$ such that $\{\ell_i\}_{i=1}^j$ is a fibrewise A_j -form of ℓ .

If there exist such $(\{\bar{m}_i^0\}, \{\ell_i^0\})$ and $(\{\bar{m}_i^1\}, \{\ell_i^1\})$, we can construct a homotopy $(\{\bar{M}_i\}, \{L_i\})$ between them similarly.

The rest of this proposition follows immediately. \square

From the above proposition and the result of the previous section, we have the following homotopy commutative diagram:

$$\begin{array}{ccccc} \Omega_0^{n-1} \text{Map}(G^{\times n}, G) & \longrightarrow & F_{A_n} G & \longrightarrow & F_{A_{n-1}} G \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_0^{n-1} \text{Map}(G_{\mathcal{P}}^{\times n}, G_{\mathcal{P}}) & \longrightarrow & F_{A_n} G_{\mathcal{P}} & \longrightarrow & F_{A_{n-1}} G_{\mathcal{P}} \end{array}$$

where the restriction $\Omega_0^{n-1} \text{Map}(G^{\times n}, G) \rightarrow \Omega_0^{n-1} \text{Map}(G_{\mathcal{P}}^{\times n}, G_{\mathcal{P}})$ is the \mathcal{P} -localization. Since $FG \rightarrow FG_{\mathcal{P}}$ is the \mathcal{P} -localization of the identity component, one can show the following corollary inductively.

Corollary 6.4. *The map $F_{A_n} G \rightarrow F_{A_n} G_{\mathcal{P}}$ induced from the map $M_n(G) \rightarrow M_n(G_{\mathcal{P}})$ above is the \mathcal{P} -localization of the identity component.*

7. FIBREWISE RATIONALIZATION OF AUTOMORPHISM BUNDLES

We review that the triviality of the fibrewise rationalization of an automorphism bundle by [4].

First, we recall the construction of classifying spaces by using the geometric bar construction given by [13] and [14]. Let G be a topological group with non-degenerate basepoint and which has the homotopy type of a CW complex. We can construct the universal bundle $EG = B(*, G, G) \rightarrow BG = B(*, G, *)$ by the geometric bar construction, where EG is also a topological group and $G \subset EG$ is a closed subgroup compatible with its right

G -action. Then BG is the coset space EG/G and the adjoint action G on EG (i.e. $(g, x) \mapsto gxg^{-1}$) induces an action of G on BG . We also call this action the adjoint action. We consider G and ΩBG as left G -space by this adjoint action.

If G is abelian, then EG is also abelian. Hence BG is again a topological abelian group. Therefore, for any abelian group Γ , the Eilenberg-MacLane space of type (Γ, n) can be taken as a topological abelian group.

For a principal G -bundle P over B , the *automorphism bundle* $\text{aut } P$ of P is the quotient space $P \times G / \sim$, where the equivalence relation \sim is defined by $(u, x) \sim (ug, g^{-1}xg)$ for any $u \in P$ and $x, g \in G$. The projection $P \rightarrow B$ induces a map $\text{aut } P \rightarrow B$ and $\text{aut } P$ is a fibre bundle with this projection. Moreover, the multiplication $G \times G \rightarrow G$ induces a fibrewise map $\text{aut } P \times_B \text{aut } P \rightarrow \text{aut } P$ and $\text{aut } P$ becomes a fibrewise topological monoid, more precisely a *fibrewise topological group*.

We show that $\text{aut } EG$ and $EG \times_G \Omega BG$ are fibrewise A_∞ -equivalent, where $EG \times_G \Omega BG$ is the quotient space with identification $(u, \ell) \sim (ug, g^{-1}\ell)$ in $EG \times \Omega BG$ and hence is also a fibrewise A_∞ -space.

We use the notation in [13] for representing elements of EG and BG . The map $\tilde{\zeta} : EG \rightarrow PBG$ is defined by $\tilde{\zeta}([g_1, \dots, g_j]g_{j+1}, t)(s) = [g_1, \dots, g_j, g_{j+1}, ((1-s)t, s)]$. Define $\zeta : G \rightarrow \Omega BG$ by the restriction of $\tilde{\zeta}$. This ζ is slightly different from May's $\tilde{\zeta}$ defined in [14] because May's ζ is not an H -map in general. By definition, the following diagram commutes:

$$\begin{array}{ccccc} G & \longrightarrow & EG & \longrightarrow & BG \\ \zeta \downarrow & & \downarrow \tilde{\zeta} & & \parallel \\ \Omega BG & \longrightarrow & PBG & \longrightarrow & BG \end{array}$$

The map ζ is G -equivariant and is a homotopy equivalence.

Lemma 7.1. *The map ζ has a G -equivariant A_∞ -form, where an A_∞ -form $\{\zeta_i\}_{i=1}^\infty$ is G -equivariant if $\zeta_i(1 \times g^{\times i}) = g \zeta_i$ for each $g \in G$ and i .*

Proof. For $g_1, \dots, g_n \in G$ and $\rho \in \mathcal{K}_i$, we have

$$\zeta(g_1, \dots, g_n)(s) = [g_1 \cdots g_n](1-s, s) = [g_1, \dots, g_n](1-s, 0, \dots, 0, s).$$

Let $\{\omega^i : \mathcal{K}_i \rightarrow P_i\}_{i=2}^\infty$ and $\{m_i : \mathcal{K}_i \times (\Omega BG)^{\times i} \rightarrow \Omega BG\}_{i=2}^\infty$ as in Example (iv) in §4. Then, for $\rho \in \mathcal{K}_n$, $g_1, \dots, g_n \in G$ and $\omega_{k-1}^n(\rho) \leq s \leq \omega_k^n(\rho)$, we obtain

$$\begin{aligned} m_i(\rho; \zeta(g_1), \dots, \zeta(g_n))(s) &= \left| [g_k], \left(1 - \frac{s - \omega_{k-1}^n(\rho)}{\omega_k^n(\rho) - \omega_{k-1}^n(\rho)}, \frac{s - \omega_{k-1}^n(\rho)}{\omega_k^n(\rho) - \omega_{k-1}^n(\rho)} \right) \right| \\ &= \left| [g_1, \dots, g_n], \left(0, \dots, 0, 1 - \frac{s - \omega_{k-1}^n(\rho)}{\omega_k^n(\rho) - \omega_{k-1}^n(\rho)}, \frac{s - \omega_{k-1}^n(\rho)}{\omega_k^n(\rho) - \omega_{k-1}^n(\rho)}, 0, \dots, 0 \right) \right|. \end{aligned}$$

From these equations, it is sufficient for us to construct a fibrewise A_∞ -form $\{\zeta_i\}_{i=1}^\infty$ such that $\zeta_n(\rho; g_1, \dots, g_n)(s) = [g_1, \dots, g_n, \xi_n(\rho; s)]$ for some $\xi_n : \mathcal{J}_n \times I \rightarrow \mathcal{A}^n$. Since each simplex \mathcal{A}^n is contractible, such $\{\xi_i\}_{i=1}^\infty$ can be constructed inductively. \square

Hence a fibrewise map $1 \times_G \zeta : EG \times_G G \rightarrow EG \times_G \Omega BG$ is a fibrewise A_∞ -equivalence.

Here, $EG \times_G \Omega BG$ is the fibrewise based loop space of the fibrewise pointed space $EG \times_G BG$ with section $BG \rightarrow EG \times_G BG$ given by $[u] \mapsto [u, e]$ for $u \in EG$ and the identity element $e \in EG$. And the map $EG \times EG \rightarrow EG \times EG$ $(u, u') \mapsto (u, uu')$ induces a fibrewise pointed topological equivalence $EG \times_G BG \rightarrow BG \times BG$, whose target is the fibrewise pointed space $BG \xrightarrow{\sigma} BG \times BG \xrightarrow{\pi} BG$ such that π is the first projection and σ is the diagonal map.

Let G be a compact connected Lie group. Since the rational cohomology ring $H^*(BG; \mathbb{Q})$ of BG is a polynomial ring, there exists the rationalization $\ell : BG \rightarrow (BG)_{(0)}$, where $(BG)_{(0)}$ is a topological abelian group. The map $1 \times \ell : BG \times BG \rightarrow BG \times (BG)_{(0)}$ is a fibrewise rationalization over BG , where $BG \times (BG)_{(0)}$ is a fibrewise pointed space with projection given by the first projection and with section $(1, \ell) : BG \rightarrow BG \times (BG)_{(0)}$. Finally, we have the fibrewise pointed topological equivalence $BG \times (BG)_{(0)} \rightarrow BG \times (BG)_{(0)}$ $(x, y) \mapsto (x, \ell(x)^{-1}y)$, whose target is a fibrewise pointed space with section given by $x \mapsto (x, e)$. Thus $EG \times_G \Omega BG \rightarrow EG \times \Omega(BG)_{(0)}$ is a fibrewise rationalization and is a fibrewise A_∞ -homomorphism. Moreover, $EG \times \Omega(BG)_{(0)}$ is trivial as a fibrewise A_∞ -space.

Theorem 7.2. *Let G be a compact connected Lie group, B be a space with homotopy type of a CW complex and P be a principal G -bundle over B . If $f : B \rightarrow M_n(G)$ is the classifying map of the automorphism bundle $\text{aut } P = P \times_G G$ of P , then the composition of f and $M_n(G) \rightarrow M_n(G_{(0)})$ is null-homotopic.*

8. PROOF OF THEOREM 1.1

We state Lemma 6.4 of [4] as the following lemma.

Lemma 8.1. *Let A_1, A_2, B_1, B_2 be groups, A_2, B_2 be abelian, A_2 be finitely generated, A_3 and B_3 be sets on which A_2 and B_2 act respectively, A_4 and B_4 be sets and $a_3 \in A_3$ be a fixed element. Consider the following commutative diagram.*

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 \\ \downarrow \ell_1 & & \downarrow \ell_2 & & \downarrow \ell_3 & & \downarrow \ell_4 \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 \end{array}$$

Assume the following conditions:

(i) The maps f_1, g_1, ℓ_1, ℓ_2 are homomorphisms, where ℓ_1 is \mathcal{Q} -surjective (i.e. for any $b \in B_1$, there exists an integer n such that $b^n \in \text{image } \ell_1$) and $\ker \ell_2$ is finite.

(ii) For any $a_2 \in A_2$ and $a \in A_3$, $\ell_3(a_2 \cdot a) = \ell_2(a_2) \cdot \ell_3(a)$.

(iii) For any $a \in A_3$, $A_2 \cdot a = f_3^{-1}(f_3(a))$. Similarly, for any $b \in B_3$, $B_2 \cdot b = g_3^{-1}(g_3(b))$.

(iv) The isotropy subgroup of a_3 is $\text{image } f_1$, and the isotropy subgroup of $\ell_3(a_3)$ is $\text{image } g_1$.

Then $f_3^{-1}(f_3(a_3)) \cap \ell_3^{-1}(\ell_3(a_3))$ is finite. Moreover, if ℓ_4 is finite-to-one and if the condition (iv) holds for any $a_3 \in A_3$, then ℓ_3 is also finite-to-one.

Proof. Fix $a \in A_2$ such that $a \cdot a_3 \in \ell_3^{-1}(\ell_3(a_3))$. From (ii) and (iii), there exists $b \in B_1$ such that $g_1(b) = \ell_2(a)$. From (i), we can take an integer n and $a' \in A_1$ such that $\ell_1(a') = b^n$. Since $\ell_2(na - f_1(a')) = 0$, the order of $na - f_1(a')$ is finite. Hence, for some integer k , $(ka) \cdot a_3 = a_3$. Then, because A_2 is finitely generated abelian group, the condition (iii) says that $f_3^{-1}(f_3(a_3)) \cap \ell_3^{-1}(\ell_3(a_3))$ is finite. The rest of this lemma follows immediately from this assertion. \square

We also quote Theorem 6.2 and Corollary 5.4 in Chapter II of [6]. Let $[X; Y]^*$ denote the homotopy set of basepoint-preserving maps from X to Y .

Theorem 8.2. *Let X be an H -space with non-degenerate basepoint and which has the homotopy type of a connected CW complex and $\ell : X \rightarrow X_{\mathcal{P}}$ be the \mathcal{P} -localization H -map. Then, for a connected finite complex W , $\ell_* : [W; X]^* \rightarrow [W; X_{\mathcal{P}}]^*$ is \mathcal{Q} -bijective, in other words, ℓ_* is \mathcal{Q} -surjective and every element in $\ker \ell_*$ has a finite order.*

Lemma 8.3. *Let G be a nilpotent finite complex and $\ell : G \rightarrow G_{\mathcal{P}}$ be the \mathcal{P} -localization. Then the homomorphism $\pi_0(FG) \rightarrow \pi_0(FG_{\mathcal{P}})$ induced from ℓ is finite-to-one.*

In the following argument, assume B is a connected finite complex, $(G, \{m_i\}_{i=2}^n)$ is an A_n -space with homotopy type of a connected finite complex such that $m_2 : G \times G \rightarrow G$ has the homotopy unit and $G_{\mathcal{P}}$ is the \mathcal{P} -localization of G .

From Corollary 6.4 and Theorem 8.2, $[\Sigma B; M_n(G)]^* \rightarrow [\Sigma B; M_n(G_{\mathcal{P}})]^*$ is \mathcal{Q} -surjective. Since G is an H -space and is a finite complex, $\pi_r(M_n(G))$ is finitely generated. Then $\pi_r(M_n(G)) \rightarrow \pi_r(M_n(G_{\mathcal{P}}))$ is finite-to-one for $r \geq 2$. In fact, this also holds for $r = 1$.

Lemma 8.4. $\pi_1(M_n(G)) \rightarrow \pi_1(M_n(G_{\mathcal{P}}))$ is finite-to-one.

Proof. Since we have the following commutative diagram, we show $\pi_0(F_{A_n}G) \rightarrow \pi_0(F_{A_n}G_{\mathcal{P}})$ is finite-to-one.

$$\begin{array}{ccc} \pi_1(M_n(G)) & \xrightarrow{\sim} & \pi_0(F_n G) \\ \downarrow & & \downarrow \\ \pi_1(M_n(G_{\mathcal{P}})) & \xrightarrow{\sim} & \pi_0(F_n G_{\mathcal{P}}) \end{array}$$

When $n = 1$, this follows by Lemma 8.3. Assume this holds for $n - 1$, $n \geq 2$. The diagram in Proposition 6.3 yields the following commutative ladder:

$$\begin{array}{ccccccc} \pi_1(F_{A_{n-1}}G) & \longrightarrow & \pi_0(\mathcal{Q}^{n-1}\text{Map}(G^{\times n}, G)) & \longrightarrow & \pi_0(F_{A_n}G) & \longrightarrow & \pi_0(F_{A_{n-1}}G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_1(F_{A_{n-1}}G_{\mathcal{P}}) & \longrightarrow & \pi_0(\mathcal{Q}^{n-1}\text{Map}(G_{\mathcal{P}}^{\times n}, G_{\mathcal{P}})) & \longrightarrow & \pi_0(F_{A_n}G_{\mathcal{P}}) & \longrightarrow & \pi_0(F_{A_{n-1}}G_{\mathcal{P}}) \end{array}$$

Let us verify the conditions of Lemma 8.1 about this diagram. The abelian group $\pi_0(\Omega^{n-1}\text{Map}(G^{\times n}, G))$ acts on $\pi_0(F_{A_n}G)$ through the above homomorphism. Similarly, $\pi_0(\Omega^{n-1}\text{Map}(G_{\mathcal{P}}^{\times n}, G_{\mathcal{P}}))$ acts on $\pi_0(F_{A_n}G_{\mathcal{P}})$. Since G is a finite complex and is an H -space, $\pi_0(\Omega^{n-1}\text{Map}(G^{\times n}, G))$ is a finitely generated abelian group. From this and Theorem 8.2, the kernel of $\pi_0(\Omega^{n-1}\text{Map}(G^{\times n}, G)) \rightarrow \pi_0(\Omega^{n-1}\text{Map}(G_{\mathcal{P}}^{\times n}, G_{\mathcal{P}}))$ is finite. The map $\pi_2(M_{n-1}(G)) \rightarrow \pi_2(M_{n-1}(G_{\mathcal{P}}))$ is \mathcal{Q} -surjective from the above, then $\pi_1(F_{A_{n-1}}G) \rightarrow \pi_1(F_{A_{n-1}}G_{\mathcal{P}})$ is \mathcal{Q} -surjective. Thus the condition (i) is verified. The conditions (ii), (iii) and (iv) (for any $a_3 \in \pi_0(F_{A_n}G)$) are easily verified. Hence $\pi_0(F_{A_n}G) \rightarrow \pi_0(F_{A_n}G_{\mathcal{P}})$ is finite-to-one. \square

Now we show the following proposition.

Proposition 8.5. $[B; M_n(G)]^* \rightarrow [B; M_n(G_{\mathcal{P}})]^*$ is finite-to-one.

Proof. When B is a point, this is trivial. When B is 1-dimensional, B is a wedge sum of finite circles. Then this proposition follows by Lemma 8.4. Assume B is a complex given by attaching the complex B' to one r -cell ($r \geq 2$), where the assertion above holds for the complex B' . Let us consider the following commutative diagram given by the cofibration $B' \rightarrow B \rightarrow S^r$:

$$\begin{array}{ccccccc} [\Sigma B'; M_n(G)]^* & \longrightarrow & \pi_r(M_n(G)) & \longrightarrow & [B; M_n(G)]^* & \longrightarrow & [B'; M_n(G)]^* \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [\Sigma B'; M_n(G_{\mathcal{P}})]^* & \longrightarrow & \pi_r(M_n(G_{\mathcal{P}})) & \longrightarrow & [B; M_n(G_{\mathcal{P}})]^* & \longrightarrow & [B'; M_n(G_{\mathcal{P}})]^* \end{array}$$

This cofibration induces the actions of $\pi_r(M_n(G))$ on $[B; M_n(G)]^*$ and $\pi_r(M_n(G_{\mathcal{P}}))$ on $[B; M_n(G_{\mathcal{P}})]^*$. Since we can apply Lemma 8.1 to this diagram, $[B; M_n(G)]^* \rightarrow [B; M_n(G_{\mathcal{P}})]^*$ is finite-to-one. \square

Here, we consider the case that G is a compact connected Lie group.

Theorem 8.6. Let B be a finite connected complex and G be a compact connected Lie group. For each $n < \infty$, the number of fibrewise A_n -equivalent classes represented by the automorphism bundles of principal G -bundle over B is finite.

Proof. From Proposition 7.2, the classifying map of the fibrewise rationalization of an automorphism bundle is null-homotopic. Then this classifying map is also null-homotopic preserving basepoint. From Proposition 8.5, there exist only finitely many classifying maps $B \rightarrow M_n(G)$ up to homotopy which correspond to some automorphism bundle. Hence the conclusion follows. \square

In general, if $(E, \{m_i\})$ is a fibrewise A_n -space, then the space of all sections $\Gamma(E)$ of E is naturally an A_n -space, where the A_n -form $\{\Gamma m_i\}$ of $\Gamma(E)$ is given by $\Gamma m_i(\rho; \varphi_1, \dots, \varphi_i)(b) = m_i(\rho; \varphi_1(b), \dots, \varphi_i(b))$. If E and E' is a fibrewise A_n -space over B and $f : E \rightarrow E'$ is a fibrewise A_n -map, then the map $\Gamma f : \Gamma(E) \rightarrow \Gamma(E')$ given by $(\Gamma f)(\varphi)(b) = f\varphi(b)$ is an A_n -map. Moreover, if f is a fibrewise A_n -equivalence, then Γf is an A_n -equivalence.

Let G be a compact connected Lie group. Since $\mathcal{G}(P)$ is isomorphic to $\Gamma(\text{aut } P)$ for any principal bundle P , Theorem 8.6 implies Theorem 1.1.

9. A COUNTEREXAMPLE WHEN $n = \infty$

In Theorem 1.1, it is essential to assume n is finite. Kono and Tsukuda have shown the following theorem:

Theorem 9.1 ([12] and [20]). Let X be an oriented simply connected closed 4-manifold and P and P' be principal $SU(2)$ -bundles over X . The classifying spaces $B\mathcal{G}(P)$ and $B\mathcal{G}(P')$ are homotopy equivalent if and only if

$$\begin{cases} |c_2(P)[X]| = |c_2(P')[X]| & \text{if } X \text{ admits an orientation reversing homotopy equivalence,} \\ c_2(P)[X] = c_2(P')[X] & \text{otherwise,} \end{cases}$$

where $c_2(P)[X]$ represents the coupling of the second Chern class of P and the fundamental class of X .

Stasheff has shown in [18] that a map between topological monoids is an A_n -map if and only if it admits an appropriate map between A_n -structures. From this, $\mathcal{G}(P)$ and $\mathcal{G}(P')$ are A_∞ -equivalent if and only if $B\mathcal{G}(P)$ and $B\mathcal{G}(P')$ are homotopy equivalent. Hence we conclude the following counterexample when $n = \infty$.

Proposition 9.2. For a simply connected closed 4-manifold X , there are infinite distinct A_∞ -types of gauge groups of principal $SU(2)$ -bundles over X .

10. THE GAUGE GROUPS OF PRINCIPAL $SU(2)$ -BUNDLES OVER S^4

In general, it is difficult problem to count the number of A_n -types of gauge groups. However, one can often partially know the behavior of the localizations of A_n -types of gauge groups.

Denote the principal $SU(2)$ -bundle over S^4 with second Chern class $k \in H^4(S^4) \simeq \mathbf{Z}$ by P_k . For a family of prime numbers \mathcal{P} , let $P_{k,\mathcal{P}}$ be the principal $SU(2)_{\mathcal{P}}$ -bundle over S^4 with second Chern class $k \in H^4(S^4; \mathbf{Z}_{\mathcal{P}}) \simeq \mathbf{Z}_{\mathcal{P}}$, where $\mathbf{Z}_{\mathcal{P}}$ is the localization of the ring \mathbf{Z} at \mathcal{P} . Here we remark that $SU(2)_{\mathcal{P}}$ can be taken as a topological group with homotopy type of a CW complex [16]. Then, using the same construction as in §7, the fibrewise \mathcal{P} -localization of $\text{aut } P_k$ as a fibrewise A_{∞} -space is $\text{aut } P_{k,\mathcal{P}}$.

Denote the identity component of the gauge group $\mathcal{G}(P_{k,\mathcal{P}})$ by $\mathcal{G}_0(P_{k,\mathcal{P}})$ and the kernel of the evaluation at the basepoint $ev : \mathcal{G}_0(P_{k,\mathcal{P}}) \rightarrow SU(2)_{\mathcal{P}}$ by $\mathcal{G}_{0,0}(P_{k,\mathcal{P}})$. Atiyah and Bott [1] constructed the universal bundle of the gauge group

$$\mathcal{G}(P_{k,\mathcal{P}}) \rightarrow E\mathcal{G}(P_{k,\mathcal{P}}) \rightarrow \text{Map}(S^4; \mathbf{HP}_{\mathcal{P}}^{\infty}; k),$$

where $E\mathcal{G}(P_{k,\mathcal{P}})$ is the space of all bundle maps from $P_{k,\mathcal{P}}$ to the universal bundle over $\mathbf{HP}_{\mathcal{P}}^{\infty}$ and $\text{Map}(S^4; \mathbf{HP}_{\mathcal{P}}^{\infty}; k)$ is the path component of $\text{Map}(S^4; \mathbf{HP}_{\mathcal{P}}^{\infty})$ corresponding to $\ell k : S^4 \rightarrow \mathbf{HP}_{\mathcal{P}}^{\infty}$. Then, the \mathcal{P} -localization of $\mathcal{G}_0(P_k)$ as an A_{∞} -space is $\mathcal{G}_0(P_{k,\mathcal{P}})$.

Definition 10.1. A fibrewise A_n -space E over B is said to be trivial if E is fibrewise A_n -equivalent to the fibrewise A_n -space $B \times E_b$ for some $b \in B$.

As in [9], $\text{aut } P_{k,\mathcal{P}}$ is trivial as a fibrewise A_n -space if and only if there exists a map $f : S^4 \times \mathbf{HP}_{\mathcal{P}}^n \rightarrow \mathbf{HP}_{\mathcal{P}}^{\infty}$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} S^4 \vee \mathbf{HP}_{\mathcal{P}}^n & \xrightarrow{\ell k \vee i} & \mathbf{HP}_{\mathcal{P}}^{\infty} \vee \mathbf{HP}_{\mathcal{P}}^{\infty} \\ \downarrow j & & \downarrow \nabla \\ S^4 \times \mathbf{HP}_{\mathcal{P}}^n & \xrightarrow{f} & \mathbf{HP}_{\mathcal{P}}^{\infty} \end{array}$$

where $k : S^4 \rightarrow \mathbf{HP}^{\infty}$ is the classifying map of P_k , $\ell : \mathbf{HP}^{\infty} \rightarrow \mathbf{HP}_{\mathcal{P}}^{\infty}$ is the \mathcal{P} -localization of \mathbf{HP}^{∞} , i and j are inclusions and ∇ is the folding map. Moreover, the following proposition holds.

Proposition 10.2. The gauge group $\mathcal{G}(P_{k,\mathcal{P}})$ of $P_{k,\mathcal{P}}$ is A_n -equivalent to the gauge group $\mathcal{G}(S^4 \times SU(2)_{\mathcal{P}})$ of the trivial bundle if and only if $\text{aut } P_{k,\mathcal{P}}$ is trivial as a fibrewise A_n -space.

Proof. We identify the gauge group $\mathcal{G}(P_{k,\mathcal{P}})$ with the space $\Gamma(\text{aut } P_{k,\mathcal{P}})$ of sections. Let $F : \mathcal{G}(S^4 \times SU(2)_{\mathcal{P}}) \rightarrow \mathcal{G}(P_{k,\mathcal{P}})$ be an A_n -equivalence and define the fibrewise A_n -map $f : B \times SU(2)_{\mathcal{P}} \rightarrow \text{aut } P_{k,\mathcal{P}}$ by $f(b, g) = f(s(g))(b)$, where $s : SU(2)_{\mathcal{P}} \rightarrow \mathcal{G}(S^4 \times SU(2)_{\mathcal{P}})$ is the standard section given by $s(g)(b) = (b, g)$. Let us see that f is a fibrewise A_n -equivalence, equivalently, $evFs : SU(2)_{\mathcal{P}} \rightarrow SU(2)_{\mathcal{P}}$ is a homotopy equivalence. Consider the following evaluation fibration:

$$\mathcal{G}_{0,0}(P_{k,\mathcal{P}}) \rightarrow \mathcal{G}_0(P_{k,\mathcal{P}}) \rightarrow SU(2)_{\mathcal{P}}.$$

Note that $\pi_i(\mathcal{G}_{0,0}(S^4 \times SU(2)_{\mathcal{P}}))$ and $\pi_i(\mathcal{G}_{0,0}(P_{k,\mathcal{P}}))$ are isomorphic finite groups for each i . Since $F_* : \pi_2(\mathcal{G}_0(S^4 \times SU(2)_{\mathcal{P}})) \rightarrow \pi_2(\mathcal{G}_0(P_{k,\mathcal{P}}))$ is an isomorphism and these groups are finite, $\pi_2(\mathcal{G}_{0,0}(S^4 \times SU(2)_{\mathcal{P}})) \rightarrow \pi_2(\mathcal{G}_{0,0}(P_{k,\mathcal{P}}))$ is also an isomorphism. Hence $ev_* : \pi_3(\mathcal{G}_0(P_{k,\mathcal{P}})) \rightarrow \pi_3(SU(2)_{\mathcal{P}})$ has a section. Then one can see that $(evFs)_* : \pi_3(SU(2)_{\mathcal{P}}) \rightarrow \pi_3(SU(2)_{\mathcal{P}})$ is an isomorphism. Therefore, $evFs : SU(2)_{\mathcal{P}} \rightarrow SU(2)_{\mathcal{P}}$ is a homotopy equivalence. \square

Proposition 10.3 ((Lemma 2.2 in [20])). If $r \in \mathbf{Z}$ is prime to every element of \mathcal{P} , then $\mathcal{G}(P_{kr,\mathcal{P}})$ is A_{∞} -equivalent to $\mathcal{G}(P_{k,\mathcal{P}})$.

Proof. The map $S^4 \rightarrow S^4$ with degree r induces the following homotopy commutative diagram:

$$\begin{array}{ccccc} \Omega_0^4 \mathbf{HP}_{\mathcal{P}}^{\infty} & \longrightarrow & \text{Map}(S^4; \mathbf{HP}_{\mathcal{P}}^{\infty}; kr) & \longrightarrow & \mathbf{HP}_{\mathcal{P}}^{\infty} \\ \downarrow & & \downarrow & & \parallel \\ \Omega_0^4 \mathbf{HP}_{\mathcal{P}}^{\infty} & \longrightarrow & \text{Map}(S^4; \mathbf{HP}_{\mathcal{P}}^{\infty}; k) & \longrightarrow & \mathbf{HP}_{\mathcal{P}}^{\infty} \end{array}$$

where $\Omega_0^4 \mathbf{HP}_{\mathcal{P}}^{\infty} \rightarrow \Omega_0^4 \mathbf{HP}_{\mathcal{P}}^{\infty}$ is a homotopy equivalence. Hence $\text{Map}(S^4; \mathbf{HP}_{\mathcal{P}}^{\infty}; kr)$ and $\text{Map}(S^4; \mathbf{HP}_{\mathcal{P}}^{\infty}; k)$ are homotopy equivalent. Therefore, $\mathcal{G}(P_{kr,\mathcal{P}})$ and $\mathcal{G}(P_{k,\mathcal{P}})$ are A_{∞} -equivalent. \square

In the rest of this section, we give an observation on the fibrewise A_n -types, using a result of [19] concerning with the homotopy groups of S^3 .

Theorem 10.4. *Let $p = \min \mathcal{P}$. If $p \geq 3$, then $\text{aut } P_{k,\mathcal{P}}$ is trivial as a fibrewise $A_{\frac{p-1}{2}-1}$ -space. Moreover, $\text{aut } P_{k,\mathcal{P}}$ is trivial as a fibrewise $A_{\frac{p-1}{2}}$ -space if and only if $k \equiv 0 \pmod{p}$.*

Proof. Let $x \in H^4(S^4; \mathbf{Z}_{\mathcal{P}})$ and $c \in H^4(\mathbf{HP}_{\mathcal{P}}^n; \mathbf{Z}_{\mathcal{P}})$ be generators such that $c|_{S^4} = x$ and $u \in H^4(K(\mathbf{Z}_{\mathcal{P}}, 4); \mathbf{Z}_{\mathcal{P}})$ be the fundamental class. From obstruction theory, we can take maps $f : S^4 \times \mathbf{HP}_{\mathcal{P}}^n \rightarrow K(\mathbf{Z}_{\mathcal{P}}, 4)$ and $g : \mathbf{HP}_{\mathcal{P}}^{\infty} \rightarrow K(\mathbf{Z}_{\mathcal{P}}, 4)$ such that $f^*u = kx \times 1 + 1 \times c \in H^4(S^4 \times \mathbf{HP}_{\mathcal{P}}^n; \mathbf{Z}_{\mathcal{P}})$ and $g^*u = c \in H^4(\mathbf{HP}_{\mathcal{P}}^{\infty}; \mathbf{Z}_{\mathcal{P}})$. Since $j^*f^*u = (\ell k \vee i)^*\nabla^*g^* = kx \vee 1 + 1 \vee c \in H^4(S^4 \vee \mathbf{HP}_{\mathcal{P}}^n; \mathbf{Z}_{\mathcal{P}})$, the following digram commutes:

$$\begin{array}{ccc} S^4 \vee \mathbf{HP}_{\mathcal{P}}^n & \xrightarrow{\ell k \vee i} & \mathbf{HP}_{\mathcal{P}}^{\infty} \vee \mathbf{HP}_{\mathcal{P}}^{\infty} \\ \downarrow j & & \downarrow \nabla \\ S^4 \times \mathbf{HP}_{\mathcal{P}}^n & & \mathbf{HP}_{\mathcal{P}}^{\infty} \\ & \searrow f & \downarrow g \\ & & K(\mathbf{Z}_{\mathcal{P}}, 4) \end{array}$$

Since $g^* : H^i(K(\mathbf{Z}_{\mathcal{P}}, 4); \mathbf{Z}_{\mathcal{P}}) \rightarrow H^i(\mathbf{HP}_{\mathcal{P}}^{\infty}; \mathbf{Z}_{\mathcal{P}})$ is an isomorphism for $i < 2p + 2$, if $n \leq (p-1)/2 - 1$, there exists a map $f' : S^4 \times \mathbf{HP}_{\mathcal{P}}^n \rightarrow \mathbf{HP}_{\mathcal{P}}^{\infty}$ such that $f'j$ and $\nabla(\ell k \vee i)$ are homotopic. This implies the first half of this theorem.

Assume that there exists a following homotopy commutative diagram:

$$\begin{array}{ccc} S^4 \vee \mathbf{HP}_{\mathcal{P}}^{(p-1)/2} & \xrightarrow{\ell k \vee i} & \mathbf{HP}_{\mathcal{P}}^{\infty} \vee \mathbf{HP}_{\mathcal{P}}^{\infty} \\ \downarrow j & & \downarrow \nabla \\ S^4 \times \mathbf{HP}_{\mathcal{P}}^{(p-1)/2} & \xrightarrow{f} & \mathbf{HP}_{\mathcal{P}}^{\infty} \end{array}$$

Then $f^*\mathcal{P}^1 u \in H^{2p+2}(S^4 \times \mathbf{HP}_{\mathcal{P}}^{(p-1)/2}; \mathbf{Z}/p\mathbf{Z})$ is computed as follows:

$$f^*\mathcal{P}^1 c = \mathcal{P}^1(kx \times 1 + 1 \times c) = k\mathcal{P}^1 x \times 1 + 1 \times \mathcal{P}^1 c = 0.$$

On the other hand, since $\mathcal{P}^1 c = \pm 2c^{(p+1)/2} \in H^{2p+2}(\mathbf{HP}_{\mathcal{P}}^{\infty}; \mathbf{Z}/p\mathbf{Z})$,

$$f^*\mathcal{P}^1 c = \pm 2f^*c^{(p+1)/2} = \pm 2(kx \times 1 + 1 \times c)^{(p+1)/2} = \pm k(p+1)x \times c^{(p-1)/2}.$$

Therefore, $k \equiv 0 \pmod{p}$.

Conversely, suppose $k \equiv 0 \pmod{p}$. Then there is some integer $r \in \mathbf{Z}$ such that $k = pr$. From the first half, we can take a map $f' : S^4 \times \mathbf{HP}_{\mathcal{P}}^{(p-3)/2} \cup * \times \mathbf{HP}_{\mathcal{P}}^{(p-1)/2} \rightarrow \mathbf{HP}_{\mathcal{P}}^{\infty}$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} S^4 \vee \mathbf{HP}_{\mathcal{P}}^{(p-1)/2} & \xrightarrow{p \vee 1} & S^4 \vee \mathbf{HP}_{\mathcal{P}}^{(p-1)/2} & \xrightarrow{\ell r \vee i} & \mathbf{HP}_{\mathcal{P}}^{\infty} \vee \mathbf{HP}_{\mathcal{P}}^{\infty} \\ \downarrow j & & \downarrow j & & \downarrow \nabla \\ S^4 \times \mathbf{HP}_{\mathcal{P}}^{(p-3)/2} \cup * \times \mathbf{HP}_{\mathcal{P}}^{(p-1)/2} & \xrightarrow{p \times 1 \cup * \times 1} & S^4 \times \mathbf{HP}_{\mathcal{P}}^{(p-3)/2} \cup * \times \mathbf{HP}_{\mathcal{P}}^{(p-1)/2} & \xrightarrow{f'} & \mathbf{HP}_{\mathcal{P}}^{\infty} \end{array}$$

Now, the obstruction to extending f' over $S^4 \times \mathbf{HP}_{\mathcal{P}}^{(p-1)/2}$ lives in $\pi_{2p+1}(\mathbf{HP}_{\mathcal{P}}^{\infty})$. Since $\pi_{2p+1}(\mathbf{HP}_{\mathcal{P}}^{\infty}) \simeq \mathbf{Z}/p\mathbf{Z}$, there is no obstruction to extending $f'(p \times 1 \cup * \times 1)$ over $S^4 \times \mathbf{HP}_{\mathcal{P}}^{(p-1)/2}$. Hence $\text{aut } P_{k,\mathcal{P}}$ is trivial as a fibrewise $A_{\frac{p-1}{2}}$ -space. \square

Furthermore, for $r \in \mathbf{Z}$ prime to each element of \mathcal{P} , we see that $\text{aut } P_{pr,\mathcal{P}}$ is not trivial as a fibrewise A_{p-1} -space.

Let ζ be the universal $SU(2)$ -bundle over \mathbf{HP}^{∞} . Then,

$$K(\mathbf{HP}_{\mathcal{P}}^{\infty})_{\mathcal{P}} = \mathbf{Z}_{\mathcal{P}}[a],$$

where $K(\cdot)_{\mathcal{P}}$ represents the \mathcal{P} -local complex K -theory and $a = \zeta - 2$. Let $b = -c_2(\zeta) \in H^4(\mathbf{HP}_{\mathcal{P}}^{\infty}; \mathbf{Q})$ then

$$cha = \sum_{j=1}^{\infty} \frac{2b^j}{(2j)!}.$$

Similarly, let $u \in \tilde{K}(S^4)_{\mathcal{P}}$ and $chu = s \in H^4(S^4; \mathcal{Q})$ such that $s = b|_{S^4}$. Assume that there exists the following homotopy commutative diagram:

$$\begin{array}{ccc} S^4 \vee \mathbf{HP}_{\mathcal{P}}^n & \xrightarrow{\ell k \vee i} & \mathbf{HP}_{\mathcal{P}}^{\infty} \vee \mathbf{HP}_{\mathcal{P}}^{\infty} \\ \downarrow j & & \downarrow \nabla \\ S^4 \times \mathbf{HP}_{\mathcal{P}}^n & \xrightarrow{f} & \mathbf{HP}_{\mathcal{P}}^{\infty} \end{array}$$

Then, $f^*b = ks \times 1 + 1 \times b$ in $H^4(S^4 \times \mathbf{HP}_{\mathcal{P}}^n; \mathcal{Q})$ and

$$f^*a = ku \times 1 + 1 \times a + \sum_{i=1}^n \epsilon_i(k)u \times a^i$$

in $\tilde{K}(S^4 \times \mathbf{HP}_{\mathcal{P}}^n)_{\mathcal{P}}$, where $\epsilon_i(k) \in \mathbf{Z}_{\mathcal{P}}$. We calculate f^*cha and $ch f^*a$ as follows:

$$\begin{aligned} f^*cha &= f^* \sum_{j=1}^{\infty} \frac{2b^j}{(2j)!} = \sum_{j=1}^{\infty} \frac{2}{(2j)!} (ks \times 1 + 1 \times b)^j = ks \times 1 + \sum_{j=1}^n \left(\frac{k}{(2j+1)!} s \times b^j + \frac{2}{(2j)!} 1 \times b^j \right), \\ ch f^*a &= ch \left(ku \times 1 + 1 \times a + \sum_{i=1}^n \epsilon_i(k)u \times a^i \right) = ks \times 1 + 1 \times \sum_{j=1}^n \frac{2}{(2j)!} b^j + \sum_{i=1}^n \sum_{j=1}^n \epsilon_i(k)s \times \left(\sum_{j=1}^n \frac{2}{(2j)!} b^j \right)^i \\ &= ks \times 1 + \sum_{j=1}^n \frac{2}{(2j)!} 1 \times b^j + \sum_{i=1}^n \sum_{l=1}^n \sum_{j_1+\dots+j_i=l} \frac{2^i \epsilon_i(k)}{(2j_1)! \cdots (2j_i)!} s \times b^l. \end{aligned}$$

Then we have the following formula:

$$\frac{k}{(2l+1)!} = \sum_{i=1}^n \sum_{\substack{j_1+\dots+j_i=l \\ j_1, \dots, j_i \geq 1}} \frac{2^i \epsilon_i(k)}{(2j_1)! \cdots (2j_i)!}.$$

From this formula, one can see that there exists the number $\epsilon_i \in \mathcal{Q}$ such that $\epsilon_i(k) = \epsilon_i k$ for each i . Of course, the sequence $\{\epsilon_i\}_{i=1}^{\infty}$ satisfy the following formula for each l :

$$\frac{1}{(2l+1)!} = \sum_{i=1}^n \sum_{\substack{j_1+\dots+j_i=l \\ j_1, \dots, j_i \geq 1}} \frac{2^i \epsilon_i}{(2j_1)! \cdots (2j_i)!}.$$

For example, $\epsilon_1 = 1/6$, $\epsilon_2 = -1/180$, $\epsilon_3 = 1/1512$ etc.

Example 3. If $\text{aut } P_k$ is trivial as a fibrewise A_3 -space, then 7560 divides k .

Proof. This condition implies that $\epsilon_1 k, \epsilon_2 k, \epsilon_3 k \in \mathbf{Z}$. Thus k is divided by $7560 = 2^3 3^3 5^1 7^1$. \square

Theorem 10.5. Let p be an odd prime. The fibrewise topological group $\text{aut } P_{k, \{p\}}$ is trivial as a fibrewise A_{p-1} -space if and only if $k \equiv 0 \pmod{p^2}$.

Proof. Assume $\text{aut } P_{k, \{p\}}$ is trivial as a fibrewise A_{p-1} -space for some $k \in \mathbf{Z}$. Then $k\epsilon_{p-1} \in \mathbf{Z}_{(p)}$. From this, to show $k \equiv 0 \pmod{p^2}$, it is sufficient to show that $p\epsilon_{p-1} \notin \mathbf{Z}_{(p)}$. Obviously, $\epsilon_1, \dots, \epsilon_{(p-3)/2} \in \mathbf{Z}_{(p)}$. Then

$$\epsilon_{(p-1)/2} = \frac{1}{p!} \pmod{\mathbf{Z}_{(p)}}.$$

From this, we have $p\epsilon_{(p+1)/2}, \dots, p\epsilon_{p-3} \in \mathbf{Z}_{(p)}$. Hence, by using the above formula for $l = p-1$, we obtain

$$0 = \frac{2^{(p-1)/2} p\epsilon_{(p-1)/2}}{(p+1)!(2!)^{(p-3)/2}} \frac{p-1}{2} + p\epsilon_{p-1} \pmod{\mathbf{Z}_{(p)}}.$$

Therefore, we get

$$p\epsilon_{p-1} = -\frac{1}{(p+1)!(p-2)!} \pmod{\mathbf{Z}_{(p)}}.$$

This implies that $p\epsilon_{p-1} \notin \mathbf{Z}_{(p)}$.

Conversely, assume $k \equiv 0 \pmod{p^2}$. Let $k' = k/p$. Since $k' \equiv 0 \pmod{p}$, we can extend $\nabla(\ell k' \vee i) : S^4 \vee \mathbf{HP}_{(p)}^{(p-1)/2} \rightarrow \mathbf{HP}_{(p)}^{\infty}$ over $S^4 \times \mathbf{HP}_{(p)}^{(p-1)/2}$ by using Theorem 10.4. Moreover, since $\pi_r(\mathbf{HP}_{(p)}^{\infty}) = 0$ for $2p+1 <$

$r < 4p - 2$, we can extend $\nabla(\ell k' \vee i) : S^4 \vee \mathbf{HP}_{(p)}^{p-2} \rightarrow \mathbf{HP}_{(p)}^\infty$ over $S^4 \times \mathbf{HP}_{(p)}^{p-2}$. Thus we can extend $\nabla(\ell k \vee i) : S^4 \vee \mathbf{HP}_{(p)}^{p-1} \rightarrow \mathbf{HP}_{(p)}^\infty$ over $S^4 \times \mathbf{HP}_{(p)}^{p-1}$ since $\pi_{4p-1}(\mathbf{HP}_{(p)}^\infty) \simeq \mathbf{Z}/p\mathbf{Z}$. \square

Now, we give the desired lower bound.

Corollary 10.6. *The number of the A_n -types of gauge groups of principal $SU(2)$ -bundles over S^4 is larger than $2^{\pi(2n+1)}$, where $\pi(m)$ represents the number of prime numbers less than or equal to m .*

Proof. Fix an odd prime p and let $k = 2^{i_2}3^{i_3}5^{i_5} \cdots p^{i_p}$ and $k' = 2^{i'_2}3^{i'_3}5^{i'_5} \cdots p^{i'_p}$, where each i_r or i'_r equals to 0 or 1. Then, from the result of [11], Proposition 10.2, Proposition 10.3 and Theorem 10.4, $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are $A_{\frac{p-1}{2}}$ -equivalent if and only if $k = k'$. Therefore, there is at least $2^{\pi(p)}$ different types of the A_n -types of the gauge groups of principal $SU(2)$ -bundles over S^4 for $n \geq (p-1)/2$. \square

Remark 6. *Similarly, from Theorem 10.5, since each two of $\text{aut } P_{0,\{p\}}$, $\text{aut } P_{1,\{p\}}$ and $\text{aut } P_{p,\{p\}}$ are not fibrewise A_{p-1} -equivalent, we have a sharper result: the number of the A_n -types of the gauge groups of principal $SU(2)$ -bundles over S^4 is larger than $2^{\pi(2n+1)-\pi(n+1)}3^{\pi(n+1)}$.*

From the prime number theorem, the number of A_n -types of the gauge groups of principal $SU(2)$ -bundles over S^4 has at least the growth of $2^{(2n+1)/\log(2n+1)}$.

This corollary gives an alternative proof of Proposition 9.2 for $X = S^4$, but does not give the complete classification of the A_∞ -types of the gauge groups.

REFERENCES

- [1] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. Lond. A **308** (1983), 523-615.
- [2] J. M. Boardman and R. M. Vogt, *Homotopy invariant structures on topological spaces*, Lecture Notes in Mathematics 347, Springer-Verlag, Berlin, 1973.
- [3] M. C. Crabb and I. M. James, *Fibrewise homotopy theory*, Springer Monographs in Mathematics, Springer-Verlag, London, 1998.
- [4] M. C. Crabb and W. A. Sutherland, *Counting homotopy types of gauge groups*, Proc. London Math. Soc. (3) **81** (2000), 747-768.
- [5] S. Forcey, *Convex hull realizations of the multiplihedra*, Topology Appl. (2) **156** (2008), 326-347.
- [6] P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, Notas de Matemàtica, North-Holland, Amsterdam, 1975.
- [7] N. Iwase and M. Mimura, *Higher homotopy associativity*, Lecture Notes in Mathematics 1370, Springer-Verlag, Berlin, 1989, 193-220.
- [8] I. M. James, *Fibrewise compactly-generated spaces*, Publ. RIMS., Kyoto Univ. **31** (1) (1995), 45-61.
- [9] D. Kishimoto and A. Kono, *Splitting of gauge groups*, Trans. Amer. Math. Soc. **362** (2010), 6715-6731.
- [10] D. A. Klarner, *Correspondences between plane trees and binary sequences*, J. Combinatorial Theory **9** (1970), 401-411.
- [11] A. Kono, *A note on the homotopy type of certain gauge groups*, Proc. Roy. Soc. Edinburgh: Sect. A **117** (1991), 295-297.
- [12] A. Kono and S. Tsukuda, *4-manifolds X over $BSU(2)$ and the corresponding homotopy types $\text{Map}(X, BSU(2))$* , J. Pure and Appl. Algebra **151** (2000), 227-237.
- [13] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics 271, Springer Berlin, 1972.
- [14] J. P. May, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc. **155** (1975).
- [15] J. P. May, *Fibrewise localization and completion*, Trans. Amer. Math. Soc. **258** (1980), 127-146.
- [16] J. Milnor, *Construction of universal bundles, I*, Ann. Math. **63** (2) (1956), 272-284.
- [17] J. D. Stasheff, *Homotopy associativity of H -spaces. I*, Trans. Amer. Math. Soc. **108** (1963), 275-292.
- [18] J. D. Stasheff, *Homotopy associativity of H -spaces. II*, Trans. Amer. Math. Soc. **108** (1963), 293-312.
- [19] H. Toda, *On iterated suspensions I*, J. Math. Kyoto Univ. **5**-1 (1965), 87-142.
- [20] S. Tsukuda, *Comparing the homotopy types of the components of $\text{Map}(S^4, BSU(2))$* , J. Pure and Appl. Algebra **161** (2001), 235-247.

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